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LOVELL'S SERIES OF SCHOOL BOOKS.

NATIONAL
ARITHMETIC,
IN
THEORY AND PRACTICE;
DESIGNED FOR THE USE OF
CANADIAN SCHOOLS.

Ermida & Elisetta Monk

BY
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PREFACE.

In preparing the following work (undertaken at the suggestion of the Chief Superintendent of Education for Upper Canada), it has been the constant aim of the Author to present it to Canadian teachers and students as a thoroughly reliable Treatise on the Theory and Practice of Numbers, and as an Arithmetic, in some degree, commensurate with the higher qualifications of teachers and the improved methods of instruction now generally found in our schools.

The Arithmetic now offered to the public is based upon the Irish National Treatise;—in fact, it was at first intended merely to adapt that work to the decimal currency, and to abbreviate the somewhat tedious reasons there given for the various rules. So many alterations and improvements suggested themselves, however, that the original design was speedily abandoned, and, with the exception of the first ten or fifteen pages, which are taken entire from the work in question, the Treatise, as at present issued, is, in all essential respects, an entirely new book. Nevertheless, as it was the sole object of the Author to prepare a *complete* text-book on the subject of Arithmetic, he has not hesitated to adopt whatever he considered good, either in the Irish National or in the numerous other excellent works on the subject.

By far the greater number of the problems are original; and it is hoped that the practical manner in which many of them are put, will tend to render the study of Arithmetic more interesting and useful than it has hitherto been. It will be observed, that a thorough series of review examples has been given at the close of each of the sections up to the seventh, and a very extensive set at the end of the book. This is deemed an important feature in the present work, as in some degree insisting upon that careful revision of what has been learned from time to time, without which, the pupil arrives at the end of the book with all the rules and principles so confounded with one another, as to render his knowledge in a great measure worthless.

Since the only difference between simple and denominate numbers is that the one increase and decrease according to the scale of tens and the other according to different scales, there is no reason why the rules relating to them should be separated; and therefore in the following pages no distinction is made between simple and compound rules. A somewhat extended experience has convinced the Author that, except to the merest beginners, the science of Arithmetic is more successfully presented by this than by the ordinary method of making the pupil learn one set of rules for simple numbers and a completely different set for compound numbers.

It will be observed that towards the end of the Treatise the rules are mainly deduced algebraically. Some teachers may not, at first, be disposed to regard this as an improvement, but it was not adopted until after careful deliberation and consultation with many of the most successful teachers of Arithmetic in the Province. It is generally conceded that a pupil should commence, in some sort, the study of Algebra as soon as he has progressed through Proportion in Arithmetic. In schools in which this view is adopted by the teacher, no difficulty can be experienced, as, even in the deduction of the rules, the algebraic principles used are of the simplest possible character.

As some teachers, however, prefer always giving the rule in a purely arithmetical form, this has invariably been appended in all the cases usually treated of in Common Arithmetic.

With regard generally to algebraic formulæ, it may be further remarked, that an algebraic formula is simply the most abbreviated form in which it is possible to express a rule or principle. Once the pupil is properly taught their use, he is in a manner independent of mere memory, since from a very few general principles he is able, without any reference to a text-book, to deduce for himself the whole series of rules for Simple and Compound Interest, Discount, Annuities, Progression, and Position. Even when the pupil is merely required to commit the rules to memory, it is obvious that he can do so much more readily when they are given to him in the shape of algebraic formulæ than in long worded paragraphs. Let any one, for instance, compare the work necessary for committing the eleven rules for Simple Interest with that required to commit the corresponding formulæ, and the result will be a thorough conviction of the superiority of the latter mode of giving the rules. In short, every experienced teacher will admit, that even while the pupil remains at school it is next to impossible to make him remember all the different rules for Interest, Progression, and Annuities; and that directly he leaves the school to enter upon the business of life, these rules are either altogether forgotten or are so confounded with one another as to become mere useless mental lumber. After many years' trial, the Author is persuaded that the only successful mode of treating the rules in question, is to enable the pupil to deduce them algebraically, and then to interpret and apply the resulting formulæ.

The attention of the teacher is respectfully directed to the Recapitulation at the end of the first section, where, it is thought, the definition and essential principles of Notation and Numeration are so concisely worded that they may be advantageously committed to memory by the pupil.

The examination questions throughout the work have been carefully prepared, and are designed both to enable the self-taught student to test, at each section, the extent and thoroughness of his knowledge of the principles therein contained, and also to guide the pupil as to what principles and definitions are of such importance that they require to be committed to memory. This latter object is further secured by the arrangement of type—all the definitions and leading principles being printed in large type, the explanations, reasons, and remarks, in small type, and the problems in a size intermediate to the two.

Great pains have been taken to render the wording of the rules as perfect as possible; and it will be observed that, in order to catch the eye when glancing over the page, they are invariably printed in Italics.

It is believed that the sections on Proportion, Fractions, Interest, &c., contain a larger amount of information, and a better selection of examples, than are commonly given; and that the section on the Properties of Numbers and the different scales of Notation will tend very materially to enlarge the pupil's acquaintance with the general principles of the science of Arithmetic.

Although the Preface is not the proper place for discussing methods of teaching Arithmetic, the Author cannot refrain from urging upon his fellow-teachers the following points:

1st. The pupil should be thoroughly drilled upon the use of the signs and symbols of Arithmetic, because these constitute the language proper to the subject.

2d. He should be required to commit to memory all the essential definitions, and also the tables of money, weights, and measures. The teacher would do well to examine his pupils on these tables once a month or oftener, since if the pupil has to turn back to his book for each table as it is required, it is not to be expected that his progress will be very rapid or thorough. It may be fairly questioned, whether more than half the difficulty and obscurity that cling to the subject of Arithmetic does not arise from the fact that the pupil is not familiar with the signs, the tables, and the principles of notation.

3d. The teacher should give his class, from time to time, questions of his own construction, either to solve at home or as ordinary school-room work, and the pupil should be encouraged and required to write questions themselves under each rule. This is an important exercise, and no teacher who once adopts it will ever throw it aside.

4th. In all operations in which there are both multiplication and division, the pupil should be taught to first indicate the processes by their appropriate signs, and then cancel as far as possible.

5th. The teacher is respectfully reminded, that without frequent and thorough reviews there can be no real progress. Experience has shown that from one-third to one-half of the time devoted to Arithmetic can be profitably devoted to revision and recapitulation.

6th. The teacher should require from his pupil the absolutely correct answer to each question. "*Near enough*" is productive of great mischief to the pupil, as it encourages a habit of such carelessness in his operations, that no confidence can be placed on his results. It is not enough that the pupil understands the principles—although this of course is important. It is possible so to train the pupil that his operations in Arithmetic shall be at once rapid and accurate, and this should be the aim of the teacher.

TORONTO, *December*, 1859.

PREFACE TO THE SECOND EDITION.

THE Author embraces the opportunity afforded by the issue of a Second Edition, both to thank his fellow-teachers in Canada for the kind and flattering reception they have given his work, and to offer a few words of explanation on what, as far as he can learn, is the only feature that does not meet with very general approval. He refers to the union of the Compound with the Simple Rules. It has been objected to the arrangement adopted in the National Arithmetic, that a pupil must know the Simple Rules before he can work problems in Reduction or in the Compound Rules. Now this is undoubtedly true, and would be a fatal objection to any such arrangement in an Elementary or Primary Arithmetic. The National is, however, an *advanced* or *second* book on Arithmetic, and the pupil is assumed to have progressed through an elementary text-book before he enters it. If the National Arithmetic were designed for beginners, where would be the necessity for a First or Elementary book on Arithmetic? The objections have arisen altogether from a misconception of the design of the book. The pupil is supposed to have worked through some elementary text-book on arithmetic, and to have acquired a certain amount of practical skill in arithmetical operations. He then commences the National, and, in progressing through it, not only meets with additional and more advanced practical exercises, but also learns the reasons and the mutual relations of the several rules. In the Elementary he is taught how to multiply an abstract by an abstract number, or an applicate by an abstract number. In the National he is shown that these operations, though differing in detail, are essentially the same in principle; and he is thus enabled to generalize and classify.

Another objection urged is, that if the National Arithmetic be designed for a second book on the science, the simple problems given at the commencement of each rule, and indeed the earlier rules themselves, should not be inserted. This is also a mistake. The object has been to exhibit a gradual progression from the simple to the more difficult—to show that the most simple and the most complicated problems depend essentially upon the same principles. Indeed, were the National Arithmetic intended merely as a second *practical* work on arithmetic, three-fourths of it might have been omitted, and nothing given but the few rules omitted in the Elementary.

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SIGNS USED IN THIS TREATISE.

$+$ the sign of addition; as $5+7$, or 5 to be added to 7.

$-$ the sign of subtraction; as $4-3$, or 3 to be subtracted from 4.

$\times$ the sign of multiplication; as 8×9, or 8 to be multiplied by 9.

$\div$ the sign of division; as $18\div6$, or 18 to be divided by 6.

( ) which is used to show that all the quantities united by it are to be considered as but one. Thus $(4+3-7)\times6$ means 4 to be added to 3, 7 to be taken from the sum, and 6 to be multiplied into the remainder. The latter is equivalent to the *whole* quantity within the brackets.

$=$ the sign of equality; as $5+6=11$, or 5 added to 6 is equal to 11.

$\frac{3}{4}>\frac{1}{2}$, and $\frac{2}{3}<\frac{8}{9}$, mean that $\frac{3}{4}$ is greater than $\frac{1}{2}$, and that $\frac{2}{3}$ is less than $\frac{8}{9}$.

: is the sign of ratio or relation; thus, $5:6$, means the ratio of 5 to 6, and is read 5 is to 6.

:: indicates the equality of ratios; thus $5:10::7:14$, means that there is the same relation between 5 and 10 as between 7 and 14; and is read 5 is to 10 as 7 is to 14.

$\sqrt{\ }$ the radical sign. By itself, it is the sign of the *square* root; as $\sqrt{5}$, which is the same as $5^{\frac{1}{2}}$, the square root of 5. $\sqrt[3]{3}$, is the cube root of 3, or $3^{\frac{1}{3}}$. $\sqrt[7]{4}$ is the 7th root of 4, or $4^{\frac{1}{7}}$, &c.

EXAMPLE. $[\sqrt{\{(8-3+7)\times4\div6\}}+31]\times\sqrt[3]{9}\div10^{\frac{1}{2}}\times5^2=556{\cdot}25$, &c., may be read thus: take 3 from 8, add 7 to the difference, multiply the result by 4, divide the product by 6, take the square root of the quotient and to it add 31, then multiply the sum by the cube root of 9, divide the product by the square root of 10, multiply the quotient by the square of 5, and the product will be equal to 556·25, &c.

These signs are *fully* explained in their proper places.

ARITHMETIC.

SECTION I.

DEFINITIONS.

1. Science is a collection of the general *principles* or leading *truths* relating to any branch of knowledge, arranged in systematic order so as to be readily remembered, referred to, and applied.

2. Art is a collection of *rules* serving to facilitate the performance of certain operations. The *rules* of Art are based upon the *principles* of Science.

3. Arithmetic is both a *Science* and an *Art.*

4. As a Science, Arithmetic treats of the nature and properties of numbers; as an Art, it teaches the mode of applying this knowledge to practical purposes. The former may be called Theoretical, and the latter Practical Arithmetic. To Practical Arithmetic belong all the *operations* we perform upon numbers, as addition, subtraction, multiplication, division, the extraction of roots, &c. The discussion of the *principles* upon which these operations are founded, constitutes the theory of Arithmetic.

5. Any single thing, as a horse, an apple, a day, an inch, is called a *unit* or *one.*

6. Numbers are expressions for one or more units, Thus, the *words one, two, three, four, five,* &c.. or the *characters* 1, 2, 3, 4, 5, &c., are expressions by which we indicate how many single things or units are to be taken.

7. Numbers are divided into two classes:
1. Abstract numbers.
2. Applicate, Concrete, or Denominate numbers.

8. If the units referred to by a number have reference to *particular* objects, as *seven days, nine inches*, &c., it is called an *applied, applicate, concrete*, or *denominate number.* If the units represented by a number have no reference to any particular object, as when we say *twice eight* are *sixteen*, or *seven* and *two* are *nine*, it is called an *abstract* number.

NOTATION AND NUMERATION.

9. To avail ourselves of the properties of numbers, we must be able both to form an idea of them ourselves, and to convey this idea to others by spoken and by written language—that is, by the voice, and by characters.

The expression of number by characters, is called *notation;* the reading of these, *numeration.* Notation, therefore, and numeration, bear the same relation to each other as *writing* and *reading*, and, though often confounded, they are in reality perfectly distinct.

10. It is obvious that, for the purposes of Arithmetic, we require the power of designating all possible numbers; it is equally obvious that we cannot give a different name, or character to each, as their variety is boundless. We must, therefore, by some means or another, make a limited system of words and signs suffice to express an unlimited amount of numerical quantities. With what beautiful simplicity and clearness this is effected, we shall better understand presently.

11. Two modes of attaining such an object present themselves; the one, that of *combining* words or characters already in use, to indicate new quantities; the other, that of representing a variety of different quantities by a *single* word or character, the danger of mistake at the same time being prevented. The Romans simplified their system of notation by adopting the principle of *combination;* but the still greater perfection of ours is due also to the expression of many numbers by the *same* character.

12. It will be useful, and not at all difficult, to explain to the pupil the mode by which, as we may suppose, an idea of considerable numbers was originally acquired, and of which, indeed, although unconsciously, we still avail ourselves; we shall see, at the same time, how methods of simplifying both numeration and notation were naturally suggested.

Let us suppose no system of numbers to be as yet constructed, and that a heap, for example, of pebbles, is placed before us that we may discover their amount. If this is considerable, we cannot ascertain it by looking at them altogether, nor even by separately inspecting them; we must, therefore, have recourse to that contrivance which the mind always uses when it desires to grasp what, taken as a whole, is too great for its powers. If we examine an extensive landscape, as the eye cannot take it all in at one view, we look successively at its different portions, and form our judgment on them in detail. We must act similarly with reference to large numbers; since we cannot comprehend them at a single glance, we must divide them into a sufficient number of parts, and, examining these in succession, acquire an indirect, but accurate idea of the whole. This process becomes by habit so rapid, that it seems, if carelessly observed, but one act, though it is made up of many; it is indispensable, whenever we desire to have a *clear idea* of numbers—which is not, however, every time they are mentioned.

13. Had we, then, to form ourselves a numerical system, we should naturally divide the individuals to be reckoned into equal groups, each group consisting of some number quite within the limit of our comprehension; if the groups were few, our object would be attained without any further effort, since we should have acquired an accurate knowledge of the number of groups, and of the number of individuals in each group, and therefore a satisfactory, although indirect estimate of the whole.

We ought to remark that different persons have very different limits to their perfect comprehension of number. The intelligent can conceive with ease a comparatively large one; there are savages so rude as to be incapable of forming an idea of one that is extremely small.

14. Let us call the *number* of individuals that we choose to constitute a group, the *ratio;* it is evident that the larger the ratio, the smaller the number of groups; and the smaller the ratio, the larger the number of groups.

15. If the groups into which we have divided the objects to be reckoned, exceed in amount that number of which we have a perfect idea, we must continue the process, and, considering the groups themselves as individuals, must form with them new groups of a higher order. We must thus proceed until the number of our highest group is sufficiently small.

16. The *ratio* used for groups of the second and higher orders, would naturally, but not necessarily, be the same as that adopted for the lowest; that is, if seven individuals constitute a group of the first order, we should probably make seven groups of the first order constitute a group of the second also; and so on.

17. It might, and very likely would happen, that we should not have so many objects as would *exactly* form a certain number of groups of the highest order—some of the next lower might be left. The same might occur in forming one or more of the other groups. We might, for example, in reckoning a heap of pebbles, have two groups of the fourth order, three of the third, none of the second, five of the first, and seven individuals or simple units.

18. If we had made each of the first order of groups consist of ten pebbles, and each of the second order consist of ten of the first, each group of the third of ten of the second, and so on with the rest, we had selected the *decimal* system, or that which is not only used at present, but which was adopted by the Hebrews, Greeks, Romans, &c. It is remarkable that the language of every civilized nation gives names to the different groups of this, but not to those of any other numerical system. Its very general diffusion, even among rude and barbarous people, has most probably arisen from the habit of counting on the fingers, which is not altogether abandoned, even by us.

19. It was not indispensable that we should have used the same *ratio* for the groups of all the different orders. We might, for example, have made four pebbles form a group of the first order, twelve groups of the first order a group of the second, and twenty groups of the second a group of the third order. In such a case we had adopted a system exactly like that to be found in the table of sterling money, in which four farthings make a group of the order of *pence*, twelve pence a group of the order of *shillings*, twenty shillings a group of the order of *pounds*. While it must be admitted that the use of the same system for applicate, as for abstract numbers, would greatly simplify our arithmetical processes—as will be evident hereafter—a glance at the tables given further on, and those set down in treating of exchange, will show that a great variety of systems have actually been constructed.

20. When we use the same *ratio* for the groups of all the orders, we term it a *common* ratio. There appears to be no particular reason why *ten* should have been selected as a "common ratio" in the system of numbers ordinarily used, except that it was suggested, as already remarked, by the mode of counting on the fingers; and that it is neither so low as unnecessarily to increase the number of orders of groups, nor so high as to exceed the conception of any one for whom the system was intended. (See Section III.)

21. A system of numbers is called *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octenary*, *nonary*, *denary*, *undenary* or *duodenary*, according as *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, *ten*, *eleven*, or *twelve*, is the *common* ratio. The *denary* and *duodenary* systems are more commonly known as the decimal and duodecimal systems. Ours is therefore a *decimal* or *denary* system of numbers.

If the common ratio were sixty, it would be a *sexagesimal* system. Such a one was formerly used, and is still, to some extent, retained—as will be perceived by the tables hereafter given for the measurement of arcs and angles, and of time. A duodecimal system would have twelve for its "common ratio"; a vigesimal, twenty, &c.

22. A little reflection will show that it was useless to give different names and characters to any numbers except to those which are less than that which constitutes the lowest group, and to the *different orders* of groups; because all possible numbers must consist of individuals, or of groups, or of both individuals and groups. In neither case would it be required to specify more than the number of individuals, and the number of each species of group, none of which numbers—as is evident—can be greater than the common ratio. This is precisely what we have done in our numerical system, except that we have formed the name of some of the groups by combining those already used. Thus, "tens of thousands," the group next higher than thousands, is designated by a combination of words already applied to express other groups—which tends still further to simplification.

23. *Arabic system of Notation*:—

	Names.	*Characters.*
Units of comparison, or simple units,	One	1
	Two	2
	Three	3
	Four	4
	Five	5
	Six	6
	Seven	7
	Eight	8
	Nine	9
First group, or units of the second order,	Ten	10
Second group, or units of the third order,	Hundred	100
Third group, or units of the fourth order,	Thousand	1,000
Fourth group, or units of the fifth order,	Ten Thousand	10,000
Fifth group, or units of the sixth order,	Hundred Thousand	100,000
Sixth group, or units of the seventh order,	Million	1,000,000

24. The characters which express the first nine numbers are the only ones used. They are called *digits*, from the custom of counting them on the fingers, already noticed,—"digitus" meaning in Latin a finger; and they have also been called *significant figures*, to distinguish them from the cipher, or 0, which has no value when standing alone, and which is used merely to give the digits their proper *position* with reference to the *decimal point*.

25. The *decimal point* is a point or dot used to indicate the position of the simple units.

The pupil will distinctly remember that the place where the "simple units" are to be found is that immediately to the left-hand of this point, which, if not expressed, is *supposed* to stand at the right-hand side of all the digits. Thus, in 468·76 the 8 expresses "simple units," being to the left of the decimal point; in 49 the 9 expresses "simple units," the decimal point being *understood* at the right of it.

26. We find by the table just given, that, after the first nine numbers, the same digits are constantly repeated, their positions with reference to the decimal point being, however, changed; that is, to indicate succeeding groups, the digit is moved, by means of a cipher, one place farther to the left. Any one of the digits may be used to express its respective number of any of the groups:—thus 8 would be eight "simple units"; 80, eight groups of the first order, or eight "tens" of simple units; 800, eight groups of the second, or units of the third order; and so on. We might use any of the digits with different groups; thus, for example, 5 for groups of the third order, 3 for those of the second, 7 for those of the first, and 8 for the "simple units," then the whole set down in full would be 5000, 300, 70, 8, or, for brevity's sake, 5378. For we *never* use a cipher, when the place it would occupy may be filled up by a digit; and it is evident that in 5378 the 378 keeps the 5 four places from the decimal point (understood), just as well as ciphers would have done; also the 78 keeps the 3 in the third, and the 8 keeps the 7 in the second place.

27. It is important to remember that each digit has two values, an *absolute* and a *relative*. The absolute value is the number of units it expresses, whatever these units may be, and is unchangeable; thus 6 always means six; sometimes, indeed, six tens; at other times six hundreds, &c. The relative value depends on the order of units indicated, and on the nature of the "simple unit." *

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* What has been said on this very important subject is intended principally for the teacher, though an ordinary amount of industry and intelligence will be quite sufficient for the purpose of explaining it, even to a child, particularly if each point is illustrated by an appropriate example; the pupil may be made, for instance, to arrange a number of pebbles in groups, sometimes of one, sometimes of another, and sometimes of several orders, and then be desired to express them by characters—the "unit of comparison" being occasionally changed from individuals, suppose to tens, or hundreds, or to scores, or dozens, &c. Indeed the pupils *must* be well acquainted with these introductory matters, otherwise they will contract the habit of answering without any very definite ideas of many things they may be called upon to explain, and which they should be expected *perfectly* to understand. Any trouble bestowed by the teacher at this period will be well repaid by the ease and rapidity with which the learner will afterwards advance. To be assured of this, he has only to recollect that most of his future reasonings will be derived from, and his explanations grounded on the very principles we have endeavoured to unfold. It may be taken as a truth, that what a child learns without understanding, he will acquire with disgust, and will soon cease to remember; for it is with children as with persons of more advanced years—when we appeal successfully to their understandings, the pride and pleasure they feel in the attainment of knowledge, cause the labour and the weariness which it costs to be undervalued or forgotten.

Pebbles will answer well for examples—indeed, their use in computing

ROMAN SYSTEM OF NOTATION.

28. Our ordinary numerical characters have not been always, or everywhere, used to express numbers; the letters of the alphabet naturally presented themselves for the purpose, as being already familiar, and, accordingly, were very generally adopted—for example, by the Hebrews, Greeks, Romans, &c., each, of course, using their own alphabet. The pupil should be acquainted with the Roman notation on account of its beautiful simplicity, and its being still employed in inscriptions, &c.: it is found in the following table:—

	Characters.	*Numbers expressed.*
	I, . . .	One.
	II, . . .	Two.
	III, . . .	Three.
Anticipated change	IIII, or IV,	Four.
Change . . .	V, . . .	Five.
	VI, . . .	Six.
	VII, . .	Seven.
	VIII, . .	Eight.
Anticipated change	IX, . . .	Nine.
Change . . .	X, . . .	Ten.
	XI, . . .	Eleven.
	XII, . .	Twelve.
	XIII, . .	Thirteen.
	XIV, . .	Fourteen.
	XV, . . .	Fifteen.
	XVI, . .	Sixteen.
	XVII, . .	Seventeen.
	XVIII, . .	Eighteen.
	XIX, . .	Nineteen.
	XX, . .	Twenty.
	XXX, &c.,	Thirty, &c.

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has given rise to the term *calculation*, "calculus" being, in Latin, a pebble; but while the teacher illustrates what he says by groups of particular objects, he must take care to notice that his remarks would be equally true of any others. He must also point out the difference between a group and its equivalent unit, which, from their perfect equality, are generally confounded. Thus, he may show that a penny, while *equal* to, is not *identical* with four farthings. This seemingly unimportant remark will be better appreciated hereafter; at the same time, without inaccuracy of result, we may, if we please, consider any group *either* as a unit of the order to which it belongs, *or* so many of the next lower as are equivalent.

	Characters.	*Numbers expressed.*
Anticipated change	XL,	Forty.
Change	L,	Fifty.
	LX, &c.,	Sixty, &c.
Anticipated change	XC,	Ninety.
Change	C,	One hundred.
	CC, &c.,	Two hundred, &c.
Anticipated change	CD,	Four hundred.
Change	D, or IↃ,	Five hundred, &c.
Anticipated change	CM,	Nine hundred.
Change	M, or CIↃ,	One thousand, &c.
	V̄, or IↃↃ,	Five thousand.
	X̄, or CCIↃↃ,	Ten thousand, &c.
	IↃↃↃↃ,	Fifty thousand, &c.
	CCCIↃↃↃ,	One hundred thousand, &c.

29. Thus we find that the Romans used very few characters—fewer, indeed, than we do, although our system is still more simple and effective from our applying the principle of "position," unknown to them.

They expressed all numbers by the following symbols, or combinations of them: IV, X, L, C, D, or IↃ, M, or CIↃ. In constructing their system, they evidently had a quinary in view; that is, as we have said, one in which five would be the *common ratio;* for we find that they changed their character, not only at ten, ten times ten, &c.; but also at five, ten times five, &c. A purely decimal system would suggest a change only at ten, ten times ten, &c.; a purely quinary, only at five, five times five, &c. As far as notation was concerned, what they adopted was neither a decimal nor a quinary system, nor even a combination of both; they appear to have supposed *two* primary groups, one of five, the other of ten "units of comparison"; and to have formed all the other groups from these, by using ten as the *common ratio* of each resulting series.

30. They anticipated a change of character,—one unit before it would naturally occur; that is, not one "simple unit," but one of the units under consideration. In this point of view, four is one unit before five; forty, one unit before fifty—tens being now the units under consideration; four hundred, one unit before five hundred—hundreds having become the units contemplated.

31. From the table (28) it will be seen that as often as any letter is repeated, so many times is its value repeated. Thus I, standing alone, denotes *one*, II denotes *two*, &c. So X denotes *ten*, XX *twenty*, &c.

When a letter of *less* value is placed *before* a letter of *greater* value, it *takes away* its own value from the greater; but when placed *after* it, it *adds* its own value to the greater. Thus V denotes *five*, IV denotes *four*, and VI *six;* so X denotes *ten*, IX *nine*, and XI *eleven*, &c.

A line or bar placed over any letter increases its value a *thousand-fold.* Thus V denotes *five*, $\overline{\text{V}}$ denotes *five thousand;* X denotes *ten*, $\overline{\text{X}}$ denotes *ten thousand*, &c.

32. To express a number by the Roman method of notation:

Rule.—*Find the highest number within the given one, that is expressed by a single character, or the "anticipation" of one* (28); *set down that character, or anticipation, as the case may be, and take its value from the given number. Find what highest number less than the remainder is expressed by a single character, or "anticipation"; put that character or "anticipation" to the right hand of what is already written, and take its value from the last remainder; proceed thus until nothing is left.*

Example.—Set down the number eighteen hundred and forty-four, in Roman characters. One thousand expressed by M is the highest number within the given one, indicated by one character or by an "anticipation"; we put down

M,

and take one thousand from the given number, which leaves eight hundred and forty-four. Five hundred, D, is the highest number within the last remainder (eight hundred and forty-four) expressed by one character, or an "anticipation"; we set down D to the right hand of M,

MD,

and take its value from eight hundred and forty-four, which leaves three hundred and forty-four. In this the highest number expressed by a single character, or an "anticipation," is one hundred, indicated by C; which we set down, and for the same reason two other C's.

MDCCC.

This leaves only forty-four, the highest number within which, expressed by a single character or an "anticipation," is forty, XL,—an "anticipation;" we set this down also,

MDCCCXL.

Four, expressed by IV, still remains; which, being also added, the whole is as follows:—

MDCCCXLIV.

Exercise 1.

33. Express the following numbers in the Roman notation:—

1. Twenty-five.
2. Forty-three.
3. Sixty-seven.
4. Eighty-nine.
5. Ninety-eight.
6. One hundred and thirty-seven.
7. Three hundred and seventy-one.
8. Four hundred and two.
9. Six hundred and seventeen.
10. Nine hundred and ninety-nine.
11. One thousand four hundred and forty-six.
12. Three thousand eight hundred and five.
13. Eight thousand six hundred and seventy.
14. Twelve thousand one hundred and sixty-nine.
15. Four hundred and ninety-seven thousand, six hundred and eighty-two.

Answers.

1. XXV. 2. XLIII. 3. LXVII.
4. LXXXIX. 5. XCVIII. 6. CXXXVII.
7. CCCLXXI. 8. CDII. 9. DCXVII.
10. CMXCIX. 11. MCDXLVI. 12. MMMDCCCV.
13. $\overline{\mathrm{V}}$MMMDCLXX. 14. $\overline{\mathrm{X}}$MMCLXIX.
15. $\overline{\mathrm{CDXCV}}$MMDCLXXXII.

Exercise 2.

34. Read the following expressions:—

1. XCVII. 2. CCLXXII. 3. DCLXVIII.
4. CMIX. 5. $\overline{\mathrm{XV}}$. 6. $\overline{\mathrm{V}}$MMMXXXIII.
7. $\overline{\mathrm{XV}}$DCCCLXXXVIII. 8. $\overline{\mathrm{DCXLV}}$MCMIV. 9. $\overline{\mathrm{XXV}}$XXV.

1. Ninety-seven.
2. Two hundred and seventy-two.
3. Six hundred and sixty-eight.
4. Nine hundred and nine.
5. Fifteen thousand.
6. Eight thousand and thirty-three.
7. Fifteen thousand eight hundred and eighty-eight.
8. Six hundred and forty-six thousand nine hundred and four.
9. Twenty-five thousand and twenty-five.

ARABIC SYSTEM OF NOTATION.

35. In the Common or Arabic system of Notation, the same character may have different values, according to the *place* it holds with reference to the *decimal point* (25), or perhaps more strictly to the simple units. This is the principle of *position.*

36. The places occupied by the units of the different orders (23), may be described as follows:—simple units, one place to the left of the decimal point, expressed, or understood; tens, two places; hundreds, three places, &c.

37. When, therefore, we are desired to write any number, we have merely to put down the digits expressing the amounts of the different units in their proper places, according to the order to which each belongs. If, in the given number, there is any "place" in which there is no digit, a cipher must be set down in that place, when required to keep another digit in its own *position.*—But a cipher produces no effect, when it is not between one or more digits and the decimal point; thus, 0536, 536·0, and 536 would mean the same thing—the first is, however, incorrect. 536 and 5360 are different; in the latter case the cipher affects the value, because it alters the *position* of the digits.

Example.—Let it be required to set down six hundred and two. The six must be in the third, and the two in the first place; for this purpose we are to put a cipher between the 6 and 2—thus, 602. Without a cipher, the six would be in the second place—thus, 62; and would mean, not six hundreds, but six tens.

38. In numerating, we begin with the digits of the highest order, and proceed downwards, stating the number which belongs to each order.

To facilitate notation and numeration, it is usual to divide the places occupied by the different orders of units into periods. For a certain distance, the English and French methods of division agree; the English billion is, however, a thousand times greater than the French. This discrepancy is not of much importance, since we are rarely obliged to use so high a number;—we shall prefer the French method. To give some idea of the amount of a billion, it is only necessary to remark, that, according to the English method of notation, there has not been one billion of seconds since the birth of Christ. Indeed, to reckon even a million, counting on an average three per second for eight hours a day, would require nearly 12 days. **The following are the two methods:**

ENGLISH METHOD.		FRENCH METHOD.
Hundreds of Thousands of Quadrillions	8	Hundreds of Octillions.
Tens of Thousands of Quadrillions	8	Tens of Octillions.
Thousands of Quadrillions	8	Octillions.
Hundreds of Quadrillions	8	Hundreds of Septillions.
Tens of Quadrillions	8	Tens of Septillions.
Quadrillions	8	Septillions.
Hundreds of Thousands of Trillions	8	Hundreds of Sextillions.
Tens of Thousands of Trillions	8	Tens of Sextillions.
Thousands of Trillions	8	Sextillions.
Hundreds of Trillions	8	Hundreds of Quintillions.
Tens of Trillions	8	Tens of Quintillions.
Trillions	8	Quintillions.
Hundreds of Thousands of Billions	8	Hundreds of Quadrillions.
Tens of Thousands of Billions	8	Tens of Quadrillions.
Thousands of Billions	8	Quadrillions.
Hundreds of Billions	8	Hundreds of Trillions.
Tens of Billions	8	Tens of Trillions.
Billions	8	Trillions.
Hundreds of Thousands of Millions	8	Hundreds of Billions.
Tens of Thousands of Millions	8	Tens of Billions.
Thousands of Millions	8	Billions.
Hundreds of Millions	8	Hundreds of Millions.
Tens of Millions	8	Tens of Millions.
Millions	8	Millions.
Hundreds of Thousands	8	Hundreds of Thousands.
Tens of Thousands	8	Tens of Thousands.
Thousands	8	Thousands.
Hundreds	8	Hundreds.
Tens	8	Tens.
Units	8	Units.

39. *Use of Periods.*—For the purpose of reading or writing numbers, we divide them by separating points, into periods—the first separating point being the decimal point, expressed or understood, and the other separating points being placed after every third digit, or place, to the right and left of the decimal point. Each period has three places—of which one or more may be occupied by digits. The lowest place in every period—or that to the *right* hand, is the "units'" place of that period: and the highest, the "hundreds'" place. And this is true, whether the period is to the left or to the right of the decimal point.

40. The period to the left of the decimal point contains the simple units. The first period to the left of the *units'* period, contains the *thousands;* and the first period to the right of it, the *thousandths.* The second period to the left of the units' period, contains the *millions;* and the second to the right of it, the *millionths.* The third period to the left of the units' period, contains the *billions;* and the third to the right of it, the *billionths.* The fourth period to the left of the units' period, contains the *trillions;* and the fourth to the right of it, the *trillionths.* The fifth period to the left of the units' period, contains the *quadrillions;* and the fifth to the right of it, the *quadrillionths.* The sixth period to the left of the units' period, contains the *quintillions;* and the sixth to the right of it, the *quintillionths.* The seventh period to the left of the units' period, contains the *sextillions;* and the seventh to the right of it, the *sextillionths.* The eighth period to the left of the units' period, contains the *septillions;* and the eighth to the right of it, the *septillionths.* The ninth period to the left of the units' period, contains the *octillions;* and the ninth to the right of it, the *octillionths.* The tenth period to the left of the units' period, contains the *nonillions;* and the tenth to the right of it, the *nonillionths.*

The pupil should be made perfectly familiar with the names of the *periods* and of the *places* in each period—so as to be able, without the slightest hesitation, to name the period and place to which any digit belongs, or into which it ought to be put. When he can read or write any *one* digit, belonging to any period and place, he should be taught to read and write a number consisting of *two, three, four,* &c., digits, whether they are close together, or separated by any number of ciphers.

The whole of what has been said above will become more evident from an attentive consideration of the following table:

of Quadrillions.			of Trillions.			of Billions.			of Millions.			of Thousands.			of Units.			of Thousandths.			of Millionths.			of Billionths.			of Trillionths.			of Quadrillionths.			of Quintillionths.		
Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units	Hundreds	Tens	Units
2	0	3	1	7	4	5	6	2	8	3	7	4	6	3	5	1	2	3	6	4	7	3	8	2	9	5	4	7	1	3	0	2	7	8	9
2	0	3,	1	7	4,	5	6	2,	8	3	7,	4	6	3,	5	1	2,	3	6	4,	7	3	8,	2	9	5,	4	7	1,	3	0	2,	7	8	9
6th Period.			5th Period.			4th Period.			3rd Period.			2nd Period.			1st Period.			1st Period.			2nd Period.			3rd Period.			4th Period.			5th Period.			6th Period.		

Examples.—Let it be required to read off the following number, 576934. We put a point to the left of the 9, and find that there are *exactly* two periods —thus, 576,934; this does not always occur, as the highest or lowest period is often imperfect, consisting only of one or two digits. Dividing the number thus into parts, shows at once that 5 is in the third place of the second period —that is, in the *Hundreds'* place of the *Thousands'* period: and therefore, that it expresses five hundred thousands: that the 7, being in the second place of the same period indicates tens of thousands: and the 6, being in the first indicates thousands. The 9, being in the third place of the first period, indicates hundreds of units: the 3, being in the second place of the same period, indicates tens of units; and the 4, being in the first, indicates units ("of comparison," or "simple units"). The number, therefore, may be read as follows—"five hundreds of thousands, seven tens of thousands, and six thousands; nine hundreds of units, three tens of units, and four units"; or more briefly, "five hundred and seventy-six thousand nine hundred and thirty-four."

41. To prevent the separating point or that which divides into periods, from being mistaken for the decimal point, the former should be a comma (,)—the latter a full stop (.) Without this distinction, two numbers which are very different might be confounded: thus, 498.763, and 498,763, one of which is a thousand times greater than the other. After a while we may dispense with the separating point, though it is convenient to retain it with large numbers, as they are then read with greater ease.

42. *To write down any integral or whole number, it is merely necessary to remember the order of the periods, and that every period contains three places, each of which must be filled, either by a digit or cipher. The one, two, or three digits, belonging to the highest period are first written in their appropriate places; then the next lower period is filled with the digits, or ciphers belonging to it; afterwards the next; and so on, till the whole number is set down.*

Example.—Let it be required to write the number seventy-three trillions two hundred and nine billions eighteen thousand and six. The highest period here mentioned is that of trillions, which we know to be the fifth to the left of the decimal point (40). We therefore set down the digits 73, bearing in mind that we are to put in four complete periods, or twelve places between the 3 and the decimal point. The next period we have is that of billions, which we fill with digits 209 (two hundred and nine). The next period, that of millions, has no significant figures, and we accordingly fill it thus, 000. We now come to the period of thousands, in which we have the digits 18, but, inasmuch as the third place of this period must also be filled, we insert there a cipher, and the full period becomes 018. Lastly, the lowest period, or that of units, is to contain only the digit 6,—the other two places being filled with ciphers, the complete period is written 006. Now setting these periods one after the other in their proper order, we obtain for the entire number the expression 73,209,000,018,006.

43. To write down any decimal number we proceed very much in the same way. We have to remark, that in any decimal the last digit to the *right* gives the denomination to the number. Thus, ·68 is read sixty-eight *hundredths*; ·4078 is read four thousand and seventy-eight *tenths of thousandths*, &c.

Now, when we wish to write any decimal, we first ascertain how many places the proposed denomination or order is to the right of the decimal point; and then, if the given digits will not bring the number to its proper position, we insert between these digits and the decimal point the requisite number of ciphers.

Example. 1.—Let it be required to write the number, seven hundred and sixteen thousand and eighty-nine billionths. Now we know (40) that billionths occupy the 9th place to the right of the decimal point. Were we to place the decimal point *immediately* before the digits themselves, thus, ·716089, they would express not so many billionths but so many millionths: since millionths occupy the 6th and billionths the 9th place. It is obvious, then, that to give the digits their proper value, we must insert three ciphers between them and the decimal point, and the number is then correctly written ·000,716,089.

EXAMPLE 2.—Write the number six thousand two hundred and one hundredths of trillionths. From (40) we know that hundredths of trillionths occupy the 14th place. The given digits (6201) being only four in number, require the aid of ten ciphers in order to fill the 14 places, and the number is thus written, ·000,000,000,062,01.

EXAMPLE 3.—Write the number six millions seven hundred and twenty-seven thousand and twelve tenths of billionths. The given digits, 6727012, are only *seven* in number, while the denomination *tenths of billionths* implies that *ten* places must be filled. We have, therefore, to insert *three* ciphers between the given digits and the decimal point, and the resulting expression, ·000,672,701,2, represents the given number.

44. The simple units are, as we have said, always found in the first period to the left of the decimal point. The digits to the left hand, progressively increase in a tenfold degree—those occupyingthe first place to the left of the simple units being ten times greater than the simple units; those occupying the second place, ten times greater than those which occupy the first, and one hundred times greater than the units of comparison themselves; and so on. Moving a digit one place to the *left*, multiplies it by ten—that is, makes it ten times greater; moving it two places, multiplies it by one hundred—that is, makes it one hundred times greater; and so of the rest. If all the digits of a quantity be moved one, two, &c., places to the left, the whole is increased ten, one hundred, &c., times—as the case may be. On the other hand moving a digit, or a quantity one place to the *right*, divides it by ten, that is makes it ten times smaller than before; moving it two places divides it by one hundred, or makes it one hundred times smaller, &c.

45. We possess this power of easily increasing, or diminishing, any number in a tenfold, &c., degree, whether the digits are all at the right, or all at the left of the decimal point; or partly at the right or partly at the left. And the pupil must remember that the quantities increase in a tenfold degree to the left, and decrease in the same degree to the right wherever the decimal point may happen to be. We therefore put quantities ten times less than simple units one place to the right of them, just as we put those which are ten times less than hundreds, &c., one place to the right of hundreds, &c. Quantities to the left of the decimal point are called *integers* because none of them is less than a *whole* simple "unit"; and those to the right of it, *decimals*. When there are decimals in a given number, the decimal point is *always* expressed, and is found at the right-hand side of the simple units.

46. The periods to the left of the decimal point may be called the *ascending*, and those to the right of it the *descending* series:—taken together, however, they constitute but *one* series, which is an ascending or a descending series, according as it is read from right to left or from left to right. Periods that are equally distant from the units of comparison bear a very close relation to each other, which is indicated even by the similarity of their names; the only difference being in the terminations (40). We have seen also, that when we divide integers into periods (40), the first separating point must be put to the right of the thousands. In dividing decimals into periods, the first point must be put to the right of the thousandths also.

47. Care must be taken not to confound what we now call "decimals," with what we shall hereafter designate "decimal fractions"; for they express equal, but not identically the same quantities—the decimals being what shall be termed the "quotients" of the corresponding decimal fractions. This remark is made here to anticipate any inaccurate idea on the subject, in those who already know something of arithmetic.

48. There is no reason for treating integers and decimals by different rules, and at different times, since they follow precisely the same laws, and constitute parts of the very same series of numbers (46). Besides, any quantity may, as far as the decimal point is concerned, be expressed in different ways; for this purpose we have merely to change the unit of comparison. Thus, let it be required to set down a number indicating five hundred and seventy-four men. If the unit be *one man*, the quantity would stand as follows, 574. If a band of *ten men*, it would become 57·4—for as each man would then constitute only the tenth part of the "unit of comparison," four men would be only four tenths, or 0·4; and since ten men would form but one unit, seventy men would be merely seven simple units, or 7, &c. Again if it were a band of one *hundred men*, the number must be written 5·74; and lastly, if a band of a *thousand men*, it would be 0·574. Should the "unit" be a band of a dozen, or a score of men, the change would be still more complicated; as, not only the position of a decimal point, but the very digits also, would be altered.

49. It is not necessary to remark that moving the decimal point so many places to the *left*, or the digits an equal number of places to the *right*, amounts to the same thing.

Sometimes in changing the decimal point, one or more ciphers are to be added; thus, when we move 42·6 three places to the left, it becomes 42600; when we move 27 five places to the right it is 0·00027, &c.

50. It follows from what we have said, that a decimal, though less than what constitutes the unit of comparison, may itself consist of not only one, but several individuals. Of course it will often be necessary to indicate the nature of the "simple units"; as 3 scores, 5 dozen, 6 men, 7 companies, 8 regiments, &c. But its nature does not affect the abstract properties of numbers; for it is true to say that seven and five, when added together, make twelve, whatever the unit of comparison may be:—provided, however, that the *same* standard be applied to both; thus 7 men and 5 men are 12 men; but 7 men and 5 horses are neither 12 men nor 12 horses; 7 men and 5 dozen men are neither 12 men nor 12 dozen men. When, therefore, numbers are to be compared, &c., they must have the same unit of comparison:—or without altering their value, they must be reduced to those which have. Thus we may consider 5 *tens* of men to become 50 *individual* men—the unit being altered from *ten men* to *one man*, without the value of the quantity being changed. This principle must be kept in mind from the very commencement, but its utility will become more obvious hereafter.

EXERCISE 8.

51. Write down the following Numbers:—

1. One hundred and ninety-four.
2. One thousand and seventy-six.
3. Twenty thousand five hundred and eight.
4. Two hundred and one thousand and three.
5. Eighty millions four thousand and thirty-three.
6. Sixteen quadrillions five hundred and ninety-seven trillions three billions forty-four millions and ninety-one.
7. Ninety-seven hundredths.
8. Six hundred and forty-three thousandths.
9. One hundred and twenty-two thousand and eighty-nine millionths.
10. Thirty-nine tenths of millionths.
11. Sixty-three hundredths of trillionths.
12. Seventeen billions four thousand and one, and nine hundred and sixty-seven billionths.
13. Seven trillions eight hundred and two billions twenty-three thousand and eleven, and nine thousand nine hundred and ninety-nine billionths.
14. One quadrillion one trillion one billion one million one thousand one hundred and one, and one trillionth.
15. Eight hundred and ninety-six trillions and two, and nine hundred and four hundredths of millionths.

Answers.

1. 194.
2. 1076.
3. 20508.
4. 201003.
5. 80004033.
6. 16597003044000091.
7. ·97.
8. ·643.
9. ·122089.
10. ·0000039.
11. ·00000000000063.
12. 17000004001·000000967.
13. 7802000023011·000009999.
14. 1001001001001101·000000000001.
15. 896000000000002·00000904.

Exercise 4.

52. Read the following numbers:—

1. 904.
2. 7060.
3. 90004.
4. 40300201.
5. 7060504030.
6. 70003000000400.
7. 604·03.
8. 90767·004003.
9. 9001·00070306.
10. 1237·9134671342913.
11. ·00100100100101.
12. 100·2003004005006007.

Answers.

1. Nine hundred and four.
2. Seven thousand and sixty.
3. Ninety thousand and four.
4. Forty millions three hundred thousand two hundred and one.
5. Seven billions sixty millions five hundred and four thousand and thirty.
6. Seventy trillions three billions and four hundred.
7. Six hundred and four, and three hundredths.
8. Ninety thousand seven hundred and sixty-seven, and four thousand and three millionths.
9. Nine thousand and one, and seventy thousand three hundred and six hundredths of millionths.
10. One thousand two hundred and thirty-seven, and nine trillion, one hundred and thirty-four billion six hundred and seventy-one million three hundred and forty-two thousand nine hundred and thirteen tenths of trillionths.
11. One hundred billion one hundred million one hundred thousand one hundred and one hundredths of trillionths.
12. One hundred, and two quadrillion three trillion four billion five million six thousand and seven tenths of quadrillionths.

ON THE DENOMINATION OF NUMBERS.

53. When two numbers have the same unit they are said to be of the same denomination; when the units are not the same, they are said to be of different denominations. For example, 16 shillings and 28 shillings are two numbers of the same denomination; but 23 shillings and three farthings are not of the same denomination, the unit of 23 shillings being one *shilling*, and of three farthings, one *farthing*. *The kind of unit always expresses the denomination.*

Even in abstract or simple numbers, different names are given to the units as we proceed to the right or left of the decimal point, viz., simple units, or units of the first order; tens, or units of the second order; hundreds, or units of the third order, &c. Considered in this relation to each other, these units may be regarded as denominate numbers.

The following Tables show the various kinds of denominate numbers in general use, and also the relative values of their different units.

TABLES OF MONEY, WEIGHTS, AND MEASURES.

STERLING MONEY.

54. The denominations are pounds, shillings, pence, and farthings.

TABLE.

4	farthings (qr.)	make	1	penny,	marked	*d.*
12	pence	"	1	shilling,	"	*s.*
20	shillings	"	1	pound,	"	£

qr.		*d.*		*s.*		£
4	=	1				
48	=	12	=	1		
960	=	240	=	20	=	1

Other English coins, some of them now out of use:—

Moidore	=	27s.	Noble	=	6s. 8d.
Guinea	=	21s.	Crown	=	5s.
Pistole	=	16s. 10d.	Angel	=	10s.
Mark or Merk	=	13s. 4d.	Groat	=	4d.

The letters £ s. d. and qr. are the initials of the Latin words, *libra*, *solidus*, *denarius*, and *quadrans*, which respectively signify a *pound*, a *shilling*, a *penny*, and a *farthing*, or quarter. The mark /, which sometimes separates the shillings and pence, is a corruption of the long ſ (s), arising from the rapidity with which it is made.

It is now customary to write farthings as fractions of a penny, as ¼d. ½d. ¾d., to represent 1 qr., 2 qr., and 3 qr.

Sterling money is supposed to have received its name from the *Esterlings* or German traders in England, by whom it is said to have been first coined.

The pound is so called, because in ancient times it was equal to a pound Troy of silver. Its present value in Canada is $4·8666, and hence the value of an English shilling is 24⅓ cents. The guinea was so called from being originally coined from gold brought from Guinea, on the coast of Africa.

The present standard gold coin of Great Britain consists of 22 parts *pure gold* and 2 parts of copper. The standard silver coin consists of 37 parts *pure* *silver* and 8 parts of copper. In copper coin 24 pence weigh a pound avoirdupois.

FEDERAL MONEY.

55. *Federal money* is the currency of the United States. The denominations are eagles, dollars, dimes, cents, and mills.

TABLE.

10 mills (m.)	make	1 cent,	marked	ct.
10 cents	"	1 dime,	"	d.
10 dimes	"	1 dollar,	"	$
10 dollars	"	1 eagle,	"	E.

m.	ct.	d.	$	E.
10 =	1			
100 =	10 =	1		
1000 =	100 =	10 =	1	
10000 =	1000 =	100 =	10 =	1.

The sign $ is the symbol for the old Spanish coin of 8 reals. On one side of the Spanish real the pillars of Hercules were represented supporting the world—on the piece of eight reals the pillars were retained and the 8 written over them—thus $. Many however consider the sign $ a contraction of the letters U. S., the initials of *United States* made by dropping the curve of the U and writing the S over it.

The present standard for both *gold* and *silver* coin in the United States is 900 parts of pure metal and 100 parts of alloy. The alloy for gold is silver and copper, of which not more than one half must be silver; that for silver is pure copper.

The gold coins are the Eagle, the Double Eagle, Half Eagle, Quarter Eagle, and Dollar; the silver coins are the Dollar, Half Dollar, Quarter Dollar, Dime, Half Dime, and three-cent piece; the copper coins are the Cent and the Half Cent; Mills are never coined.

OLD CANADIAN MONEY.

56. The denominations are pounds, dollars, shillings, pence, and farthings.

TABLE.

4 farthings	make	1 penny,	marked	d.
12 pence	"	1 shilling,	"	s.
5 shillings	"	1 dollar,	"	$
4 dollars	"	1 pound,	"	£

qr.	d.	s.	$	£
4 =	1			
48 =	12 =	1		
240 =	60 =	5 =	1	
960 =	240 =	20 =	4 =	1.

NOTE.—Every 3d. of the old coinage is equal to 5 cents of the new. The York shilling is equal to the eighth part of a $, or to 7½d. or to 12½ cents.

NEW CANADIAN OR DECIMAL MONEY.

57. The denominations are dollars and cents.

The coins are cents, five-cent pieces, ten-cent pieces, and twenty-cent pieces.

100 cents (c) make 1 dollar, marked $

AVOIRDUPOIS WEIGHT

58. Is used in weighing heavy articles. Its name is derived from French—and ultimately from Latin words signifying "to have weight." Its denominations are tons, hundredweights, quarters, pounds, ounces, and drams.

TABLE.

16 drams	make	1 ounce,	marked oz.
16 ounces	"	1 pound,	" lb.
25 pounds	"	1 quarter,	" qr.
4 quarters	"	1 hundredweight,	" cwt.
20 cwt.	"	1 ton,	" t.

d.		oz.		lb.		qr.		cwt.		t.
16	=	1								
256	=	16	=	1						
6400	=	400	=	25	=	1				
25600	=	1600	=	100	=	4	=	1		
512000	=	32000	=	2000	=	80	=	20	=	1.

It was formerly the custom to allow 28 lbs. to the quarter, 112 lbs. to the hundredweight, and 2240 to the ton. This has now fallen into disuse: and among merchants in Canada the qr., cwt., and ton are universally considered as respectively equal to 25 lbs., 100 lbs., and 2000 lbs. The Custom Houses continue to regard the cwt. as equal to 112 lbs., and some few articles are still weighed by the old cwt. by farmers and others. The English cwt. is 112 lbs.

TROY WEIGHT.

59. The denominations of Troy Weight are pounds, ounces, pennyweights, and grains.

TABLE.

24 grains (grs.)	make	1 pennyweight,	marked dwt.
20 pennyweights	"	1 ounce,	" oz.
12 ounces	"	1 pound,	" lb.

grs.		dwt.		oz.		lb.
24	=	1				
480	=	20	=	1		
5760	=	240	=	12	=	1.

This weight was introduced into Europe from Cairo, in Egypt, and was first adopted in Troyes, a city of France—whence its name. It is used in philosophy, in weighing gold, precious stones, &c.

NOTE.—The origin of all weights used in England, was a grain of wheat taken from the middle of the ear and well dried. A weight equal to 32 of these grains was called a *pennyweight*, being equal to the weight of a silver penny then in use; 20 of these pennyweights constituted an ounce, which was the 12th part of a pound (Lat. "uncia," a 12th part—compare "*inch*," the twelfth part of a foot). In later times the pennyweight came to be divided into 24 equal parts instead of 32, but these still retain the name of grains.

The "Carat," which is equal to about four grains (somewhat less than Troy grains), is used in weighing diamonds. The term carat is also applied in estimating the fineness of gold: the latter, when perfectly pure, is said to be "24 *carats* fine." If there are 23 parts gold, and one part some other material, the mixture is said to be "23 carats fine"; if 22 parts out of the 24 are gold, it is "22 carats fine," &c. The whole mass is in all cases supposed to be divided into 24 parts, of which the number consisting of gold is specified. Our gold coin is 22 carats fine; pure gold, being very soft, would too soon wear out. The degree of fineness of gold articles is marked upon them at the Goldsmiths' Hall; thus we generally perceive "18" on the cases of gold watches; this indicates that they are "18 carats fine"—the lowest degree of purity which is stamped.

	grs.
A Troy ounce contains	480
An Avoirdupois ounce	437½
A Troy pound	5,760
An Avoirdupois pound	7,000

A Troy pound is equal to 372·965 French grammes.

175 Troy pounds are equal to 144 avoirdupois; 175 Troy are equal to 192 avoirdupois ounces.

APOTHECARIES' WEIGHT.

60. The denominations of Apothecaries' Weight are pounds, ounces, drams, scruples, and grains.

TABLE.

20 grains (grs.)	make	1 scruple,	marked	sc. or ℈
3 scruples	"	1 dram,	"	dr. or ʒ
8 drams	"	1 ounce,	"	oz. or ℥
12 ounces	"	1 pound,	"	lb.

grs.		℈		ʒ		℥		lb.
20	=	1						
60	=	3	=	1				
480	=	24	=	8	=	1		
5760	=	288	=	96	=	12	=	1.

Apothecaries mix their medicines by this weight, but buy and sell by avoirdupois.

The pound and ounce of this weight are the same as in Troy weight.

LONG MEASURE.

61. The denominations of Long Measure are leagues, miles, furlongs, rods, yards, feet, inches, and lines.

TABLE.

12 lines (l.)	make	1 inch,	marked	in.
12 inches	"	1 foot,	"	ft.
3 feet	"	1 yard	"	yd.
5½ yards	"	1 rod, pole, or perch,		rd. or p.
40 rods or perches	"	1 furlong	"	fur.
8 furlongs	"	1 mile,	"	m.
3 miles	"	1 league,	"	lea.
69⅙ miles (nearly)	"	1 degree or 360th part of the earth's circumference.		

in.		ft.		yd.		rd.		fur.		m.
12	=	1								
36	=	3	=	1						
198	=	16½	=	5½	=	1				
7920	=	660	=	220	=	40	=	1		
63360	=	5280	=	1760	=	320	=	8	=	1.

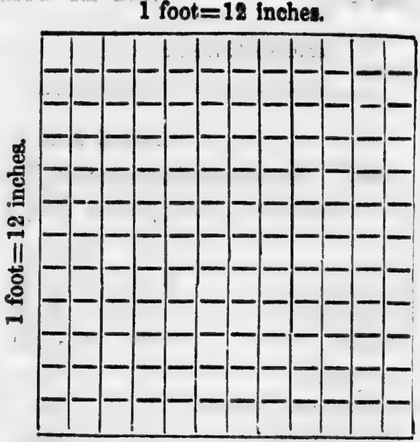
100 links, 4 rods, or 22 yards, make 1 Gunter's chain. Each link therefore is equal to 7$\frac{92}{100}$ inches.

Eleven Irish are equal to 14 English miles. The Paris foot is equal to 12·792 English inches, the Roman foot to 11·604 English inches, and the French metre to 39·383 English inches.

4 inches	make	1 hand (used in measuring horses).
8 inches	"	1 palm.
18 inches	"	1 cubit.
3 feet	"	a common pace.
5 feet	"	a Roman pace.
6 feet	"	a fathom.
120 fathoms	"	a cable's length.

SQUARE MEASURE.

62. This measure is used for estimating artificers' work, such as flooring, plastering, painting, paving, &c., and, in short, any kind of work where surface alone is concerned. It is always employed in measuring land, and hence it is frequently called Land Measure.



A square is a four sided figure having all of its sides equal and perpendicular one to another. If the length of each side be an inch, a foot or a yard, &c., the square is called a square inch, a square foot, or a square yard, &c. It will be observed from the adjacent figure that a square foot contains 12 × 12 or 144 square inches, and similarly a square yard may be shown to contain 3 × 3 or 9 square feet.

The denominations of Square Measure are square miles, acres, rods, square perches, square yards, square feet, and square inches.

TABLE.

144 square inches	make	1 square foot,	marked	sq. ft.
9 square feet	"	1 square yard,	"	sq. yd.
30¼ square yards	"	1 square rod,	"	sq. rd.
40 square rods	"	1 rood,	"	r.
4 roods	"	1 acre,	"	a.
640 acres	"	1 square mile,	"	s. m.

sq. in.		sq. ft.		sq. yd.		sq. rd.		r.		acre
144	=	1								
1296	=	9	=	1						
39204	=	272¼	=	30¼	=	1				
1568160	=	10890	=	1210	=	40	=	1		
6272640	=	43560	=	4840	=	160	=	4	=	1

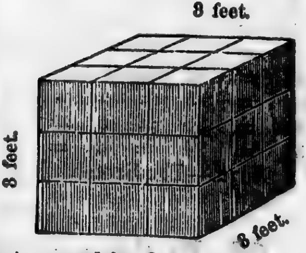
63. In measuring land, Gunter's chain is used. It is divided into 100 links.

$7\frac{92}{100}$ inches	make	1 link,	marked	l.
100 links or 4 rods	"	1 chain,	"	c.
80 chains	"	1 mile,	"	m.
10000 square links	"	1 square chain,	"	sq. c.
10 square chains	"	1 acre	"	a.

SOLID OR CUBIC MEASURE.

64. This measure is used for finding the solid contents of timber, stone, &c. A cube is a solid bounded by six equal surfaces or squares, and having eight equal edges. It is called a cubic inch, a cubic foot, or a cubic yard, according as each of these edges is an inch, a foot, or a yard in length.

The accompanying figure represents a cubic yard—each edge being 3 feet in length. The top, which is equal to the base, contains 3 × 3 or 9 square feet; hence, if it were only one foot in height it would contain 9 cubic feet; but it is 3 feet in height, and must therefore contain 9 × 3 or 27 cubic feet. A cubic yard then contains 3 × 3 × 3 or 27 cubic feet.



Similarly it may be shown that a cubic foot contains 12×12×12 or 1728 cubic inches.

The denominations of Cubic Measure are cords, tons, cubic feet, and cubic inches.

TABLE.

1728 cubic inches make 1 c. ft. marked c. ft.
27 cubic feet " 1 cubic yd. " c. yd.
*40 c. ft. of round timber, or 50 c. ft. of sq. or hewn timber } " 1 ton, " ton.
128 cubic feet make 1 cord of firewood, marked c.

c. in.		c. ft.		c. yd.
1728	=	1		
46656	=	27	=	1.

A pile of cord-wood 4 feet high, 4 feet wide, and 8 feet long, contains 128 cubic feet or one cord. One foot in length of such a pile is called a *cord-foot.* It is equal to 16 solid feet, and is consequently equivalent to the eighth part of a cord.

CLOTH MEASURE.

65. The denominations of Cloth Measure are French ells, English ells, Flemish ells, quarters, nails and inches.

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* A ton of round timber is that quantity of timber which, when hewn, will make 40 cubic feet.

TABLE.

2¼ inches (in.)	make	1 nail,	marked	na.
4 nails	"	1 quarter,	"	qr.
3 quarters	"	1 Flemish ell,	"	Fl. e.
4 quarters	"	1 yard,	"	yd.
5 quarters	"	1 English ell,	"	E. e.
6 quarters	"	1 French ell,	"	F. e.

in.	na.	qr.	Fl. e.	yd.	Eng. e.	Fr. e.
2¼	1					
9	4	1				
27	12	3	1			
36	16	4	1⅓	1		
45	20	5	1⅔	1¼	1	
54	24	6	2	1½	1⅕	1.

NOTE.—The Scotch ell contains 4 quarters 1⅕ inch.

DRY MEASURE.

66. By this are measured all dry wares, as grain, beans, coal, oysters, &c.

The denominations of Dry Measure are chaldrons, bushels, pecks, gallons, quarts, and pints.

TABLE.

2 pints (pt.)	make	1 quart,	marked	qt.
4 quarts	"	1 gallon,	"	gal.
2 gallons	"	1 peck,	"	pk.
4 pecks	"	1 bushel,	"	bu.
36 bushels	"	1 chaldron,	"	ch.

pt.	qt.	gal.	pk.	bu.	ch.
2	1				
8	4	1			
16	8	2	1		
64	32	8	4	1	
2304	1152	288	144	36	1.

Our Standard of Dry Measure is the Winchester bushel. This is an upright cylinder whose internal diameter is 18½ inches and depth 8 inches. It contains 2150·4 cubic inches or 77·627 lbs. Avoirdupois of pure distilled water at 62° Fahr. and 30 in. barometer. The standard unit of Dry Measure in the United States is also the Winchester bushel, so called because the standard measure was formerly kept at Winchester, England. The standard unit of Dry Measure in Great Britain is the Imperial bushel, which is an upright cylinder whose internal diameter is 18·789 inches and depth 8 inches. It contains 2218·192 cubic inches or 80 lbs. Avoirdupois of pure distilled water at 62° Fahr. and 30 in. barometer.

Grain is often bought and sold by weight, allowing for a bushel, 60 lbs. of wheat, 56 lbs. of rye, 56 lbs. of Indian corn 48 lbs. of barley, 34 lbs. of oats, 60 lbs. of peas, 50 lbs. of beans, 40 lbs. of buckwheat, 60 lbs. of timothy or red clover seed.

LIQUID MEASURE.

67. Liquid Measure is used for measuring all liquids.

The denominations of Liquid Measure are tuns, pipes, hogsheads, barrels, gallons, quarts, pints, and gills.

TABLE.

4 gills (g.)	make	1 pint,	marked	pt.
2 pints	"	1 quart,	"	qt.
4 quarts	"	1 gallon,	"	gal.
31½ gallons	"	1 barrel,	"	bar.
2 barrels	"	1 hogshead,	"	hhd.
2 hogsheads	"	1 pipe,	"	pi.
2 pipes	"	1 tun;	"	tun.

g.	pt.	qt.	gal.	bar.	hhd.	pi.	tun.
4 =	1						
8 =	2 =	1					
32 =	8 =	4 =	1				
1008 =	252 =	126 =	31½ =	1			
2016 =	504 =	252 =	63 =	2 =	1		
4032 =	1008 =	504 =	126 =	4 =	2 =	1	
8064 =	2016 =	1008 =	252 =	8 =	4 =	2 =	1

The English Imperial gallon contains 277·274 cubic inches or 10 lbs. avoirdupois of pure distilled water, weighed at a temperature of 62° Fahr. and under a barometric pressure of 30 inches.

In the United States the wine gallon contains 231 cubic inches, and the beer gallon 282 cubic inches. The gallon of Great Britain is therefore about equal to 1·2 gallons United States Wine Measure.

By an Act of the Imperial Parliament, 1826, the Imperial gallon of 277·274 cubic inches, was adopted as the only gallon, and is therefore the standard for both liquid and dry measure.

Beer is usually sold by the gallon; sometimes, however, in casks of 5 gals., 10 gals., 20 gals., &c. The beer barrel contains 36 gallons, and the hogshead 54 gallons.

TIME MEASURE.

68. Time is naturally divided into days and years—the former measured by the revolution of the earth on its axis, and the latter by the revolution of the earth round the sun.

The denominations of Time Measure are years, months, weeks, days, hours, minutes, and seconds.

TABLE.

60 seconds (sec.) make 1 minute, marked min.
60 minutes " 1 hour, " h.
24 hours " 1 day, " d.
7 days " 1 week, " wk.
4 weeks " 1 lunar month, " mo.
13 lunar months or
12 calendar months or } make 1 civil year, marked yr.
365$\frac{1}{4}$ days (nearly)

sec.		min.		h.		da.		wk.		yr.
60	=	1								
3600	=	60	=	1						
86400	=	1440	=	24	=	1				
604800	=	10080	=	168	=	7	=	1		
31557600	=	525960	=	8766	=	365$\frac{1}{4}$	=	52$\frac{5}{28}$	=	1.

The twelve calendar months, into which the civil or legal year is divided, and the number of days in each, are as follows.

First	month,	January,	has	31	days.
Second	"	February	"	28	"
Third	"	March	"	31	"
Fourth	"	April,	"	30	"
Fifth	"	May,	"	31	"
Sixth	"	June,	"	30	"
Seventh	"	July,	"	31	"
Eighth	"	August,	"	31	"
Ninth	"	September,	"	30	"
Tenth	"	October,	"	31	"
Eleventh	"	November,	"	30	"
Twelfth	"	December,	"	31	"

The number of days in the respective months may be recalled by recollecting the following well-known lines:

Thirty days hath September,
April, June, and November;
February has twenty-eight alone,
And all the rest have thirty-one;
But leap-year coming once in four,
February then has one day more.

The number of days in each month may also be recollected by counting the months on the *four* fingers and *three* intervening spaces. Thus, January on the first finger; February in space between first and second fingers; March on second finger; April in second space; May on third finger; June in third space; July on fourth finger; August on first finger (since there are no more spaces); September in first space, &c. Now, when counted thus, all the months having 31 days come on the fingers, and all having 30 only fall into the spaces.

The solar year is the time elapsing from the passage of the sun from either solstice back to the same again, and is equal to 365d. 5h. 48m. 48sec.

The sidereal year is the time between two successive conjunctions of the sun with some star, and is equal to 365d. 6h. 9m. 14$\frac{1}{2}$sec.

The civil or legal year is that in common use among different nations, and is equal to 365 days for three years in succession and to 366 days for the fourth.

This additional day is given to every fourth year, in order to make the civil year agree with the solar. It was originally added by repeating the *sixth* of the calends of March in the Roman calendar—corresponding with the 24th of February with us. The day was called the *intercalary* day, from the Latin *intercalo*, to insert; and the year was called *bissextile*, from the Latin *bis*, twice, and *sextilis*, sixth (i. e., *sixth* calend, taken *twice*). We now call it Leap Year, because it leaps a day more than a common year. This correction was made by Julius Cæsar, emperor of Rome, and hence the civil year is often called the Julian year.

The addition of one day every four years would be strictly correct, if the solar year contained 365d. 6h.; but it only contains 365d. 5h. 48m. 48s., or 11m. 12s. less than 365d. 6h. Adding 1 day every 4 years, gives us then an error of excess of 44m. 48s., or about 3 days for every 400 years. Thus the Julian calendar was behind the solar time, since the Julian year was longer than the natural year. This error, at the time of Pope Gregory XIII., amounted to 10 days, which he corrected in 1582 by suppressing 10 days in the month of October, the day after the 4th being called the 15th. Hence this calendar is sometimes called the *Gregorian calendar*.

This correction was not adopted in England till 1752, when the error amounted to 11 days. By Act of Parliament, 11 days after the 2nd of September were therefore omitted. The *civil* year, by the same act, was made to commence on the 1st of January, instead of the 25th of March, as it had done previously.

Dates reckoned by the *old method* or Julian calendar, are called *Old Style;* and those reckoned by the *new method* are called *New Style.*

To change any date from Old to New Style, we must add 11 days to it; and if the given date in Old Style is between the 1st of January and the 25th of March, we must add 1 to the year in New Style.

Russia still reckons dates according to Old Style. The difference now amounts to 12 days.

69. To ascertain whether a year is LEAP YEAR.

Divide the given year by 4, and if there is no remainder it is Leap Year. The remainder, if any, shows how many years have elapsed since a Leap Year occurred.

Thus, dividing the year 1847 by 4, the remainder is 3; hence it is 3 years since the last Leap Year, and the ensuing year will be Leap Year.

To this rule there is an exception; for we have seen that a *solar* year is 11m. 12s. less than a Julian year, which is 365¼ days. This error, in 400 years, amounts to about 3 days; consequently if a day is added every *fourth* year, that is, if we have 100 leap years in 400 years, according to the Julian calendar, the reckoning would fall 3 days behind the *solar* time. Thus reckoning from the commencement of the Christian era, when it was January 1st, 401, by the Julian time, it was January 4th by the *solar* time.

To remedy this error, only 1 *centennial* year in 4 is regarded as leap year; or, which is the same in effect, whenever the *centennial* year, or the *number* expressing the century, is not divisible by 4, that year is not a leap year, while the other centennial years are. Thus, 17, 18, 19, denoting 1700, 1800, and 1900, are *not divisible* by 4, consequently they are *not* leap years, though according to the rule above they would be; on the other hand, 16 and 20, denoting 1600 and 2000, *are divisible* by 4, and are therefore leap years. There is still a slight error, but it is so small that in 5000 years it scarcely amounts to a day.

70. TABLE SHOWING THE NUMBER OF DAYS FROM ANY DAY OF ONE MONTH TO THE SAME DAY OF ANY OTHER MONTH IN THE SAME YEAR.

From any day of	To the same day of											
	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

The months counted from any day of, are arranged in the left-hand vertical column; those counted to the same day of, are in the upper horizontal line; the days between these periods are found in the angle of intersection, in the same way as in a common table of multiplication. If the end of February be included between the two points of time, a day must be added in leap years.

Example 1.—How many days are there from the 15th of March to the 4th of October? Looking down the vertical row of numbers at the head of which October is placed, and at the same time along the horizontal row at the left hand side of which is March, we perceive in their intersection the number 214: so many days, therefore, intervene between the 15th of March to the 15th of October. But the 4th of October is 11 days earlier than the 15th: we therefore subtract 11 from 214, and obtain 203, the number required.

Example 2.—How many days are there between the 3rd of January and the 19th of May? Looking as before in the table, we find that 120 days intervene between the 3rd of January and the 3rd of May; but as the 19th is 16 days later than the 3rd, we add 16 to 120, and obtain 136, the number required.

Since February is in this case included, if it were a leap year, as that month would then contain 29 days, we should add 1 to the 136, and 137 would be the answer.

EXAMPLES.

1. How many days from May 3rd to the 4th of next July?

Ans. 62 days.

2. How many days from July 4th to the 25th of next December?

Ans. 174 days.

3. How many days from March 21st to the 23rd of the next September?

Ans. 186 days.

4. How many days from September 23rd to the 21st of the next March? *Ans.* 179 days.

5. How many days from June 21st to the 22nd of the next December? *Ans.* 184 days.

6. How many days from December 22nd to the 21st of the next June? *Ans.* 181 days.

7. How many days from March 21st to the 21st of the next June? *Ans.* 92 days.

8. How many days from January 13th, 1848, to September 17th of the same year? *Ans.* 248 days.

71. The *unit of time* is the *basis* of that of *Length*, *Mass*, and *Pressure*: the connections being as follows:—

A pound pressure means that amount of pressure which is exerted towards the earth, at the level of the sea, by the *quantity of matter* called a *pound*.

A pound of Matter means a quantity equal to that quantity of pure water which, at the temperature of 62° Fahr., would occupy 27·272 *cubic inches*.

A cubic inch is that cube whose side, taken 39·1393 times, would measure the effective length of a London *seconds-pendulum*.

A London seconds-pendulum is that which, by the unassisted and unopposed effect of its own gravity, would make 86400 vibrations in an artificial solar day, or 86163·09 in a natural sidereal day.

CIRCULAR MEASURE.

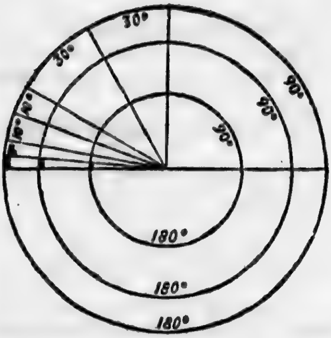
72. Circular Measure, sometimes called Angular Measure, is chiefly used by astronomers, navigators, and surveyors, for measuring angles and for reckoning *latitude* and *longitude*, and the motion of the heavenly bodies.

The denominations of Circular Measure are signs, degrees, minutes, and seconds.

TABLE.

60 seconds (″)	make	1 minute,	marked	′
60 minutes	"	1 degree,	"	°
30 degrees	"	1 sign,	"	s.
12 signs or 360 deg.		1 circle,	"	c.

″		′		°		s.		c.
60	=	1						
3600	=	60	=	1				
108000	=	1800	=	30	=	1		
1296000	=	21600	=	360	=	12	=	1.



The circumference of every circle is supposed to be divided into 360 equal parts called degrees, as in the subjoined figure. Since a degree is simply the $\frac{1}{360}$ part of the circumference of the circle, it is obvious that its length must depend upon the size of the circle. If the circumference be 360 miles in length, then a degree of that circle will be one *mile* long; if the circle be 360 inches in circumference, then a degree will be one *inch*, &c.

The division of the circumference of the circle into 360 equal parts took its origin from the length of the year, which, in round numbers, was supposed to contain 360 days, or 12 months of 30 days each. The 12 *signs* correspond to the 12 months.

The term *minute* is from the Latin *minutum* "a small part." The term seconds is an abbreviated expression for *second* minutes, or minutes of the *second order*.

MISCELLANEOUS TABLE.

73.	12 individual things	make	1 dozen.
	12 dozen.............	"	1 gross.
	12 gross..............	"	1 great gross.
	20 individual things	"	1 score.
	24 sheets of paper..	"	1 quire.
	20 quires.............	"	1 ream.
	112 pounds............	"	1 quintal.
	200 "	"	1 barrel of pork or beef.
	196 "	"	1 barrel of flour.
	14 "	"	1 stone.

BOOKS.

A sheet folded into two leaves is called a *folio*.
" folded into four leaves is called a *quarto*, 4to.
" folded into eight leaves is called an *octavo*, or 8vo.
" folded into twelve leaves is called a *duodecimo*, or 12mo.
" folded into eighteen leaves is called an 18mo.

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74. When figures are written by the side of each other, thus,

2587931272.

the language implies that the *unit* in each place is equivalent to ten units of the place next to the right; or that ten units of any particular place are equivalent to one unit of the place immediately to the left.

75. When figures are written thus,

$	d.	c.	m.
1	4	6	5

the language implies that 10 units of the lowest denomination make one of the second; ten of the second, one of the third; and ten of the third, one of the fourth.

76. When figures are written thus,

T.	cwt.	qr.	lb.	oz.	dr.
16	11	3	21	14	8

the language implies that 16 units of the lowest denomination make one of the second; 16 units of the second, one of the third; 25 units of the third, one of the fourth; 4 of the fourth, one of the fifth; and 20 of the fifth, one of the sixth.

All other denominate numbers are formed on the same principle; and in all of them we pass from a lower to the next higher denomination by considering how many units of the one make one unit of the other.

REDUCTION.

77. Reduction is the changing the denomination of a number from one unit to another, without altering the value of the number. For example, if we desire to reduce 7 of the order of *hundreds* to a lower denomination, we multiply the 7 by 10, and thus obtain 70 of the order *tens*, which are equal to 7 of the *third order* or *hundreds*. If we wish to reduce to a still lower denomination, we multiply the *tens* by ten, and this gives us 700 of the *first order* or *simple units*, which are just equal to 70 *tens* or 7 *hundreds*.

If, on the contrary, we wish to reduce 900 of the *first order* or *simple units*, to units of the *third order* or *hundreds*, we divide by 10, and thus obtain 90 of the *second order*, which we again divide by 10 and obtain 9 units of the *third order* or *hundreds*.

Hence reduction of denominate numbers is divided into two parts:—

1st. To reduce a number from a higher denomination to a lower: this is called Reduction Descending.

2nd. To reduce a number from a lower denomination to a higher: this is called Reduction Ascending.

REDUCTION DESCENDING.

EXAMPLE.

78. Reduce £6 16s. 0¼d. to farthings.

```
   £   s.  d.
   6  16  0¼
  20
----
 136 shillings = £6 16s.
  12
----
1632 pence = £6 16s. 0d.
   4
----
6529 farthings = £6 16s. 0¼d.
```

Explanation.—In this example we multiply the £6 by 20, *because* each pound is equal to 20 shillings; 6 pounds are therefore equal to 120 shillings, and the 16 shillings given in the question make 136 shillings. Then we multiply the number of shillings by 12, *because* each shilling is equal to 12 pence, and, since there are no pence in the question, we simply set down the result, 1632 pence. Lastly, we multiply the 1682 pence by 4, *because* each penny is equal to 4 farthings, and to the result we add the one farthing given in the question.

From the above example and solution we deduce the following—

RULE.

Multiply the highest given denomination by that quantity which expresses the number of the next lower contained in one of its units; and add to the product that number of the next lower denomination which is found in the quantity to be reduced.

Proceed in the same way with the result; and continue the process until the required denomination is obtained.

Exercise 5.

1. How many farthings in 23328 pence? *Ans.* 93312.
2. How many shillings in £348? *Ans.* 6960.
3. How many pence in £38 10*s.*? *Ans.* 9240.
4. How many pence in £58 13*s.*? *Ans.* 14076.
5. How many farthings in £58 13*s.*? *Ans.* 56304.
6. How many farthings in £59 13*s.* 6¾*d.* *Ans.* 57291.
7. How many pence in £63 0*s.* 9*d.*? *Ans.* 15129.
8. How many pounds in 16 cwt., 2 qrs., 16 lb.? *Ans.* 1666.
9. How many pounds in 14 cwt., 3 qrs., 16 lb.? *Ans.* 1491.
10. How many grains in 3 lb., 5 oz., 12 dwts., 16 grains? *Ans.* 19984.

11. How many grains in 7 lb., 11 oz., 15 dwt., 14 grains? *Ans.* 45974.

12. How many hours in 20 (common) years? *Ans.* 175200.

13. How many feet in 1 mile? *Ans.* 5280.

14. How many minutes in 46 years, 21 days, 8 hours, 56 minutes (not taking leap-year into account)? *Ans.* 24208376.

15. How many square yards in 74 square perches? *Ans.* 2238·5 (2238 and a half).

16. How many square yards in 46 acres, 3 roods, 12 perches? *Ans.* 226633.

17. How many square acres in 767 square miles? *Ans.* 490880.

18. How many cubic inches in 767 cubic feet? *Ans.* 1325376.

19. How many quarts in 767 pecks? *Ans.* 6136.

20. How many pints in 797 pecks? *Ans.* 12752.

REDUCTION ASCENDING.

79. Example.—Reduce 856347 farthings to pounds, &c.

4)856347

12)214086¾d.

20)17840s. 6¾d.

£892 0s. 6¾d. = 856347 farthings.

Explanation.—We divide the farthings by 4, *because* every four farthings are equal to one penny, and it is evident that what remains after taking away four farthings as often as possible from the farthings must be farthings. We thus obtain 856347 farthings, equal to 214086 pence and 3 farthings. Then we divide the pence by 12, *because* every 12 pence are equivalent to one shilling, and what remains after taking 12 pence as often as possible from the pence must be pence. We thus ascertain that 214086 pence and 3 farthings are equal to 17840 shillings and 6 pence 3 farthings. Lastly we divide 17840 shillings by 20, *because* every 20 shillings are equal to one pound. By this process we have reduced 856347 farthings to £892 0s. 6¾d.

From the above example and solution we deduce the following—

RULE.

Divide the given number by that number which it takes of the given denomination to make one of the next higher. Set down the remainder, if any, and proceed in the same manner with each successive denomination till you come to the one required. The last quotient, with the several remainders annexed, will be the answer required.

Exercise 6.

1. Reduce 32756 farthings to pounds, shillings, and pence. *Ans.* £34 2s. 5d.

2. Reduce 23547 troy grains to pounds, &c. *Ans.* 4 lb. 1 oz. 1 dwt. 3 grs.

3. Reduce 397024 yards to miles, furlongs, &c.
Ans. 225 m. 4 fur. 26 r. 1 yd.
4. How many hours are there in 28635 seconds?
Ans. 7 h. 57 min. 15 sec.
5. How many cwt., qrs., and pounds in 1666 pounds?
Ans. 16 cwt., 2 qrs. 16 lb.
6. How many cwt., &c. in 1491 pounds?
Ans. 14 cwt. 3 qrs. 16 lb.
7. How many pounds troy in 115200 grains? *Ans.* 20.
8. How many pounds in 107520 oz. avoirdupois? *Ans.* 6720.
9. How many cubic feet, &c. in 1674674 cubic inches?
Ans. 969 feet, 242 inches.
10. How many yards in 767 Flemish ells?
Ans. 575 yards, 1 quarter.
11. How many leagues in 183810 feet?
Ans. 11 lea. 1 m. 6 fur. 20 rd.
12. How many cubic yards in 138297 cubic inches?
Ans. 2 c. yds. 26 ft. 57 in.
13. How many cords of wood are there in 67893 cubic feet?
Ans. 530 cords, 53 cub. ft.
14. In 3561829 seconds, how many weeks?
Ans. 5 wks. 6 dys. 5 h. 23 min. 49 sec.
15. In 1597 quarts, how many bushels?
Ans. 49 bushels, 3 pks. 1 gal. 1 qt.
16. In 1000 cord-feet of wood, how many cords?
Ans. 125 cords.
17. In 10,000″ how many degrees? *Ans.* 2° 46′ 40″
18. In 70,000 square links, how many square chains?
Ans. 7 square chains.
19. In 11521 grains apothecaries' weight, how many pounds?
Ans. 2 lbs. 0℥ 0ʒ 0℈ 1 gr.
20. In 26025 square feet, how many roods?
Ans. 2 r. 15 sq. p. 17 sq. yds. 8 sq. ft. 36 sq. in.

REDUCTION OF THE OLD CANADIAN CURRENCY TO THE NEW OR DECIMAL CURRENCY.

30. **Example.**—Reduce £76 14s. 10¾d. to cents.

£76 × 400	=	30400 cents.
14s. × 20	=	280 "
10¾d. = 43 far. × 5 ÷ 12	=	$17\frac{11}{12}$ "
£76 14s. 10¾d.	=	$30697\frac{11}{12}$ cts.

Explanation. — We multiply £76 by 400, *because* each pound is equal to 4 dollars or 400 cents; next we multiply 14, the number of shillings, by 20, *because* each shilling is equal to 20 cents; lastly we multiply the number of farthings in the pence and farthings by 5 and divide the result by 12, *because* each farthing is equal to $\frac{5}{12}$ of a cent.

That each farthing is equal to $\frac{5}{12}$ of a cent is evident from the fact that

48 farthings (or one shilling) are equal to 20 cents; or 12 farthings equal 5 cents, or one farthing equal $\frac{5}{12}$ of a cent.

From the above example and solution we deduce the following—

RULE.

Multiply the pounds by 400, the shillings by 20, and take five-twelfths of the number expressing how many farthings there are in the given pence and farthings. Add the three results together and their sum will be the number of cents required.

Consider the last two figures as cents, and the result will be dollars and cents.

Note.—We take five-twelfths of the farthings by multiplying them by five and dividing the result by twelve.

Exercise 7.

1. How many cents are there in £3 7s. $1\frac{1}{4}$d.? *Ans.* $1842\frac{1}{12}$ cents.
2. How many dollars are there in £29 18s. $3\frac{1}{2}$d.?
Ans. $11965\frac{5}{6}$ cents, or \$$119·65\frac{5}{6}$ cents.
3. How many cents are there in $11\frac{1}{4}$d.? *Ans.* $18\frac{3}{4}$ cents.
4. How many dollars and cents are there in £69 15s. 6d.?
Ans. 27910 cents, or \$279·10.
5. How many dollars and cents in 18s. $8\frac{1}{2}$d.? *Ans.* \$$3·74\frac{1}{6}$.
6. How many dollars and cents in £17 16s. $5\frac{3}{4}$d.?
7. How many dollars and cents in £87? *Ans.* \$$71·29\frac{7}{12}$.
8. How many dollars and cents in 15s. $11\frac{3}{4}$d.? *Ans.* \$348·00.
9. How many dollars and cents in £16 6s. 2d.? *Ans.* \$$3\ 19\frac{7}{12}$.
10. Reduce £2 9s. 11d. to dollars and cents. *Ans.* \$$65·23\frac{1}{3}$.
Ans. \$$9·98\frac{1}{3}$.

RECAPITULATION.

I. Science is a collection of the general *principles* or *leading truths* of any branch of knowledge systematically arranged.

II. Art is a collection of *rules* serving to facilitate the performance of certain operations.

III. The rules of art are based upon the principles of science.

IV. Arithmetic is both a science and an art.

V. The science of arithmetic discusses the *properties* of numbers and the *principles* upon which the elementary operations of arithmetic are founded.

VI. The *science of arithmetic* is called Theoretical Arithmetic.

VII. The *art of arithmetic* is called Practical Arithmetic.

VIII. *Practical Arithmetic* is the *application* of *rules* based upon the science of numbers, to practical purposes, as the solution of problems, &c.

IX. *Numbers* are expressions for one or more things of the same kind.

X. *Unity*, or the *unit* of a number, is one of the equal things which the number expresses.

XI. Numbers are divided into two classes, viz.: simple or abstract numbers; and applicate, concrete, or denominate numbers.

XII. An applicate, concrete, or denominate number is a number whose unit indicates some particular object or thing.

XIII. A simple or abstract number is a number whose unit indicates no particular object or thing.

XIV. Numbers may be expressed either by words or by characters.

XV. The expression of numbers by characters is called *Notation.*

XVI. The reading of numbers, expressed by characters, is called *Numeration.*

XVII. The characters *we* use to express numbers are either *letters* or *figures.*

XVIII. The expression of numbers by *letters* is called Roman Notation.

XIX. The expression of numbers by *figures* is called Arabic Notation.

XX. In the Roman Notation only seven numeral letters are used, viz.: I, V, X, L, C, D, M.

XXI. When these letters stand alone, I denotes *one*, V *five*, X *ten*, L *fifty*, C *one hundred*, D *five hundred*, M *one thousand.*

XXII. All other numbers are expressed by repetitions and combinations of these letters.

XXIII. In combinations of these numerical letters, every time a letter is repeated its value is repeated; also when a letter of a lower value stands *before* one of a higher, its value is to be *subtracted;* but when a letter of a lower comes directly *after* one of a higher value, its value is to be added.

XXIV. A bar or dash written over a letter or combination of letters, multiplies the value by one thousand. As we have already a character for one thousand, viz., M, and can, by repeating it, express *two* or *three thousand*, we do not dash the I, or combinations into which it enters.

XXV. Anciently, IV was written IIII; IX was written VIIII; XL was written XXXX, &c.; D was written Iↄ, and M was written CIↄ. Affixing C to Iↄ increases its value ten times—thus Iↄ=500; Iↄↄ=5000; Iↄↄↄ=50000, &c. Prefixing C and affixing ↄ to CIↄ increases its value also ten times, thus CIↄ=1000; CCIↄↄ=10000; CCCIↄↄↄ=100,000, &c.

XXVI. The figures or characters used in the Arabic or common system of notation are 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, one, two, three, four, five, six, seven, eight, nine, zero.

XXVII. The first nine of these characters are called *significant figures*, because each one has always some value or denotes some number. They are also called *digits* (Lat. *digitus*, "a finger"), from the almost universal habit of counting on the *fingers*.

XXVIII. The last or *zero* is called a *cipher* or *naught*, because it is *valueless*, that is, stands for *nothing*. It is not, however, *useless*, since it serves to give the significant figures their appropriate places.

XXIX. When the 0 stands to the left of an integral number or to the right of a decimal, i. e. when it does not come *between* the decimal point and some significant figure, it is both *valueless* and *useless*.

XXX. The digits 1, 2, 3, &c. standing immediately to the left of the *decimal* point expressed or understood, are called *simple units*, or *units of the first order*.

XXXI. The *decimal point* is a small dot or point, used to indicate the *position* of the *simple units*.

XXXII. The digits 1, 2, 3, &c. standing one place to the left of the simple units, are called *tens*, or units of the *second order to the left*. When they stand one place to the right of the simple unit, they are called *tenths*, or units of the *second order to the right*.

XXXIII. The digits 1, 2, 3, &c. when standing two places to the left of the simple unit, are called *hundreds*, or units of the *third order to the left.* When standing two places to the right, they are called *hundredths*, or units of the *third order to the right*, &c.

XXXIV. Commencing at the simple units and proceeding to the *left*, we have *units* of the *first order* or *simple units;* next, units of the *second order* or *tens;* next, units of the *third order* or *hundreds;* next, units of the *fourth order* or *thousands;* next, units of the *fifth order* or *tens of thousands*, &c.

XXXV. Commencing at the simple units and proceeding to the *right*, we have units of the *first order* or *simple units;* next, units of the *second order* or *tenths;* next, units of the *third order* or *hundredths;* next, units of the *fourth order* or *thousandths;* next, units of the *fifth order* or *tenths of thousandths*, &c.

XXXVI. Each digit has two values, viz.: a simple or absolute value, and a local or relative value.

XXXVII. The *simple* or *absolute value* of a digit is the value it expresses when simply considered as representing a certain number of repetitions of the digit *one.*

XXXVIII. The *local* or *relative* value of a digit is the value it expresses when considered as occupying a certain *position* with reference to the decimal point.

XXXIX. The *ratio* of one number to another is the *relation* which one bears to the other with respect to magnitude, when the comparison is made by considering, not by *how much* the one is greater or less than the other, but what *number of times* it contains it, or is contained in it.

XL. When several numbers, or groups of units, are so arranged that the second and third have the same ratio to one another as the first and second, and the third and fourth the same ratio as the second and third, &c.,—they (the numbers or groups of units) are said to have a *common ratio.*

XLI. The common ratio of our system of numbers is 10—by saying which we merely mean that the different orders increase or decrease from one another in a ten-fold proportion, i. e. that 10 units of any one order make one unit of the next higher, and *vice versâ.*

XLII. A system of numbers is called a *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octenary*, *nonary*, *denary*, &c. system, according as *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, or *ten* is the common ratio of the orders. Ours is a denary or decimal system.

XLIII. To facilitate the reading of a number we divide it into *periods* of three places each, by placing separating points after every third figure right and left of the decimal point.

XLIV. The periods to the left of the decimal point are *units*, *thousands*, *millions*, *billions*, *trillions*, &c. The periods to the right of the decimal point are *thousandths*, *millionths*, *billionths*, *trillionths*, &c.

XLV. The *lowest order* used in any reading, whether it be thousands, units, hundredths, tenths of thousandths, hundredths of millionths, &c., gives the name or denomination to the part or whole of the number used in the reading.

XLVI. Numbers to the left of the decimal point are *integers* or whole numbers; those to the right of the decimal point are called *decimals.*

XLVII. A number is multiplied by 10 every time the decimal point is moved *one* place to the right, and divided by 10 every time the decimal point is moved *one* place to the left. Thus, moving the decimal point *two*, *four*, or *six* places, either multiplies or divides the number by 100, 10,000, or 1,000,000, according as we move it to the right or to the left.

XLVIII. A number may be read in several ways by changing the nature of the simple unit. Thus the number 576·24 may be read:

1st. Five hundreds, seven tens, six units, two tenths, and four hundredths.
2nd. Fifty-seven tens, six units, two tenths, and four hundredths.
3rd. Five hundred and seventy-six units, two tenths, and four hundredths.
4th. Five thousand, seven hundred and sixty-two tenths, and four hundredths.
5th. Fifty-seven thousand, six hundred and twenty-four hundredths.

6th. Five hundred and seven thousand, six hundred and twenty-four hundredths.
7th. Fifty-seven tens, and six hundred and twenty-four hundredths.
8th. Five hundred and seventy-six units, and twenty-four hundredths.
9th. Fifty-seven tens, sixty-two tenths, and four hundredths.
10th. Five hundreds, seven hundred and sixty-two tenths, and four hundredths, &c.

Exercise 8.

MISCELLANEOUS PROBLEMS.

1. Reduce 6789634 links to acres, and prove by reducing the result to links.

2. Read 67845398678904 and 59007040600400000·00060604.

3. Set down 4769 in Roman numerals.

4. Make 42986 ten thousand times greater.

5. Reduce £16 18s. 6¾d. Old Canadian Currency to Dollars and Cents.

6. Read $\overline{\text{LXXV}}$MMCMXCI.

7. Write down, in Arabic numerals, six hundred and five billions, seventy thousand and sixteen, and nine millionths.

8. Make 469789 one hundred times greater.

9. Read the number 6798 in *all* the ways it can be read. (See Recapitulation XLVIII.)

10. Divide 69800463 by *one million*.

11. Divide 8439 by ten thousand.

12. Multiply 6789 by one hundred thousand.

13. Multiply 60432986 by ten millions.

14. Write down one quadrillion one billion one thousand and one, and one trillionth.

15. Write down seven thousand six hundred and nine tenths of millionths.

16. Read 90807060504030 and
400404040040000000060432·01010203040506.

17. Reduce 6789463 inches to acres, and prove by reducing the result to inches.

18. Reduce 617 cord-feet of wood to cords.

19. Reduce 91867 cubic feet of wood to cords.

20. Write down 718, 614, 499, 999, 8643, 96149, 163986, and 444444 in Roman numerals.

21. Read CCCXXXIII, MCMLXXXIX, and $\overline{\text{M}}$I.

22. Read 6129 in as many ways as it can be read.

23. Give all the readings of 634986.

24. Give all the readings of 19·639.

25. Reduce 18s. 9¼d.; £6 2s. 11d.; 3s. 7d.; and £189 7s. 4¾d. to dollars and cents.

26. Give all the readings of the number $69·863 Federal money.

27. Give all the readings of 9 bush. 3 pk. 1 gal. 3 qts. 1 pt.

28. Were the years 1693, 1856, 1728, 1549, 867, 444, 1600, and 927, leap years or not? If not, how many years after or before leap year?

29. How many days from this to the 17th of next March?

30. Answer the following questions: What is the meaning of the symbols £ s. d. and q.? In the expression "$^{18}/_{9}$" what does the long mark (/) represent? What is the derivation of the word sterling? Why are the pound and guinea so called? What is the derivation of the sign $? What is the derivation of the words "grain," "pennyweight," "ounce," and "inch"? What is a "carat"? What is a square? Show that a square yard contains 9 square feet. Show that a cubic yard contains 27 cubic feet. What is a cubic yard? What is meant by a ton of round timber? What must be the dimensions of a pile of wood in order that it shall contain a cord? What is meant by a *cord-foot?* What are the dimensions of the *Imperial-bushel?*—of the *Winchester-bushel?* Which of these is our standard? Which that of the United States? How many pounds of wheat go to the bushel?—of rye?—of oats?—of barley?—of peas?—of beans?—of buckwheat?—of Indian corn? What is our standard for liquid measure? How many cubic inches of water are there in the Imperial gallon? How many pounds Avoirdupois? What are the standard gallons of the United States? Explain why a day is added to every fourth year. What is the origin of the divisions of the circle into degrees and signs? What is the derivation of the terms "minute" and "second"? How many sheets of paper are there in a quire? How many quires in a ream? How many pounds are there in a barrel of flour? What is the meaning of folio?—of 4to or quarto?—of 8vo or octavo?—of 12mo or duodecimo?—of 16mo?—of 18mo?

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*Numbers in Roman numerals, thus, XVI, refer to the articles in the recapitulation: those in Arabic numerals, thus,* 16, *refer to the numbered articles of the Section.*

1. What is science? (I.)
3. Upon what are the rules of art based? (III.)
5. What are the objects of the science of arithmetic? (V.)
7. What name is given to the art of arithmetic? (VII.)
9. What are numbers? (IX.)
11. How many classes of numbers are there? (XI.)
13. What are simple or abstract numbers? (XIII.)
2. What is art? (II.)
4. Is arithmetic a science or an art? (IV.)
6. What is the science of arithmetic called? (VI.)
8. What is practical arithmetic? (VIII.)
10. What is the unit of a number? (X.)
12. What are applicate or denominate numbers? (XII.)
14. By how many methods may numbers be expressed? (XIV.)
15. What is Notation? (XV.)
16. What is Numeration? (XVI.)
17. What characters do *we* use to express numbers? (XVII.)
18. What is Roman Notation? (XVIII.)
19. What is Arabic Notation? (XIX.)
20. What numeral letters are used in Roman Notation? (XX.)
21. What is the value of each of these letters when standing alone? (XXI.)
22. How are all other numbers expressed in Roman Notation? (XXII.)
23. In combination, when a letter is repeated, what does it indicate? (XXIII.)
24. When a letter of a lower is placed before one of a higher value, what does it indicate? (XXIII.)
25. When a letter of a lower is placed after one of a higher value, what does it indicate? (XXIII.)
26. What effect has a bar or dash written over a letter or expression? (XXIV.)
27. How do we always write 1000, 2000, 3000? (XXIV.)
28. Why do we not dash the I or expressions into which it enters? (XXIV.)
29. How were *four*, *nine*, *forty*, &c., anciently written? (XXV.)
30. How were 500 and 1000 anciently written? (XXV.)
31. How were the expressions IↃ and CIↃ increased in value in ten-fold proportion? (XXV.)
32. What are the characters used in Arabic or Common Notation? (XXVI.)
33. What are significant figures, and why are they so called? (XXVII.)
34. What are digits, and why are they so called? (XXVII.)
35. Why is 0 called "cipher" or "naught"? (XXVIII.)
36. Is the cipher of any value? Is it of any use? (XXVIII.)
37. When is the cipher or 0 *both valueless* and *useless*? (XXIX.)
38. When are digits called *simple units* or units of the first order? (XXX.)
39. What is the decimal point? (XXXI.)
40. When are digits called *tens* or units of the second order to the left? (XXXII.)
41. When are digits called *tenths* or units of the second order to the right? (XXXII.)
42. When are digits called hundreds, thousands, hundredths, thousandths, &c.? (XXXIII.)
43. Name the different orders to the left of the decimal point,—and to the right. (XXXIV.) (XXXV.)
44. How many values has each digit? What are they? (XXXVI.)
45. What is the simple or absolute value of a digit? (XXXVII.)
46. What is the local or relative value of a digit? (XXXVIII.)
47. What is meant by the ratio one number bears to another? (XXXIX.)
48. What is meant by a common ratio. (XL.)
49. What is meant by saying that 10 is the common ratio of our *system of numbers?* (XLI.)

50. What name is given to a system having 10 for its common ratio?—to one having 6?—to one having 8?—to one having 2?—to one having 12?—to one having 7? (XLII.)
51. Why are periods used? How many places are there in each period? (XLIII.)
52. Name the periods right and left of the decimal point. (XLIV.)
53. What order gives the name or denomination to the number read? (XLV.)
54. What are *integers*? What are *decimals*? (XLVI.)
55. How does it affect a number to remove the decimal point to the right? How to remove it to the left? (XLVII.)
56. How may a number be read in several ways? (XLVIII.)
57. When figures are written thus, 673·82 what does the notation imply? (74.)
58. When figures are written thus, 6d. 23h. 16 min. 37 sec., what does the notation imply? (75 and 76.)
59. What is Reduction? (77.)
60. Into what two parts is Reduction divided? (77.)
61. What is Reduction Descending? Give an example. (77.)
62. What is Reduction Ascending? Give an example. (77.)
63. Give the rule for Reduction Descending. (78.)
64. Give the Rule for Reduction Ascending. (79.)
65. What are the denominations of Sterling money? Give the table. (54.)
66. How are pounds, shillings, and pence reduced to farthings? Give the process and the reason for each step (54 and 78.) (Answer this and similar succeeding questions after the following *model.*) We multiply the pounds by twenty, and add in the shillings *because* each pound is equal to twenty shillings We multiply the shillings by twelve and add in the pence, *because* each shilling is equal to twelve pence. And lastly, we multiply the pence by four and add in the farthings, *because* each penny is equal to four farthings.
67. What are the denominations of Federal money? Give the table. (55.)
68. What are the denominations of Canadian money, old currency? Give the table. (56.)
69. What are the denominations of Canadian money, new currency? Give the table (57.)
70. How is Old Canadian Currency reduced to New? Give the process and reasons for each step. (80.)
71. What are the denominations of Avoirdupois weight? Give the table. (58.)
72. How many pounds are there in the new cwt.? How many in the old cwt.? (58.)
73. How are tons reduced to drams? (58 and 78.)
74. What are the denominations of Troy weight? Give the table. (59.)
75. How are grains Troy reduced to pounds Troy? Give the process and reason for each step. (59 and 79.) (Answer this and succeeding similar questions after the following *model.*) We divide the grains by 24, *because* every 24 grains are equal to one pennyweight. We divide the resulting pennyweights by 20, *because* every 20 pennyweights are equal to one ounce. And lastly, we divide the resulting ounces by 12, *because* every 12 ounces are equal to one pound.
76. What are the denominations of Apothecaries' weight? Give the table. (60.)
77. How are pounds, ounces, &c., Apothecaries' weight reduced to grains? (60 and 78.) Answer as in question 66.
78. What are the denominations of Long measure? Give the table. (61.)
79. How are lines reduced to leagues? (61 and 79.) Answer after model in question 75.
80. What are the denominations of Square measure? Give the table. (62.)
81. How are square miles reduced to square inches? (62 and 78.) Answer after model.
82. How are links reduced to acres? (63 and 79.) Answer after model.
83. What are the denominations of Solid measure? Give the table. (64.)
84. How are cubic inches reduced to cubic feet? (64 and 79.)
85. How are cubic feet of wood reduced to cords? (64 and 79.)
86. What is a *cord-foot*? (64.)

87. What are the denominations of Cloth measure? Give the table. (65.)
88. How are English ells reduced to inches? (65 and 78.) Answer after model.
89. What are the denominations of Dry measure? Give the table. (66.)
90. How are pints reduced to chaldrons? (66 and 79.) Answer after model.
91. What are the denominations of Liquid measure? Give the table. (67.)
92. How are tuns reduced to gills? (67 and 78.) Answer after model.
93. What are the denominations of Time measure? Give the table. (68.)
94. How are seconds reduced to years? (68 and 79.) Answer after model.
95. Name the months and the number of days in each. (68.)
96. What is the Solar year and its length?—the Sidereal year and its length?—the Civil year and its length? (68.)
97 How can we ascertain whether any given year be Leap year? (69.)
98. Show that the unit of time is the basis of the units of length, mass or capacity, and weight. (71.)
99. What are the denominations of Circular measure? Give the table. (72.)
100. Upon what does the length of a degree depend? (72.) How are degrees reduced to seconds? (72 and 78.)

SECTION II.

FUNDAMENTAL RULES.

1. Arithmetic may be divided into four parts:—

1st. The Arithmetic of Whole Numbers, or that which treats of the properties of entire units.

2nd. The Arithmetic of Fractions, or that which treats of the parts of units.

3rd. The Arithmetic of Ratios, which treats of the relations of numbers, whether integral or fractional, to each other and to the unit 1.

4th. The Application of Arithmetic to practical and useful purposes.

2. The Arithmetic of Whole numbers includes Addition, Subtraction, Multiplication, Division, Involution, Evolution, &c.

3. The Arithmetic of Fractions may be divided into two parts:—

1st. Vulgar or Common Fractions, in which the unit is *divided into any number of equal parts.*

2nd. Decimal Fractions in which the unit is divided according to the *scale of ten.*

4. The Arithmetic of Ratios relates to the comparison of numbers with respect to their quotients, and embraces Proportion and Progression.

5. Addition, Subtraction, Multiplication, Division, are called the *fundemental rules,* or *ground rules* of Arithmetic, because all the other operations of Arithmetic are performed by means of them.

6. Whatever operations we may perform upon a number, we can only either *increase* it or *diminish* it. If we increase it, the process belongs to addition; if we diminish it, to subtraction. All the rules of Arithmetic are therefore resolvable into these two. Multiplication is only a short method of performing a peculiar kind of addition, in which the addends are all the same; and division is merely an abridged method of performing a particular kind of subtraction, in which the same quantity is to be taken away from a given number as often as possible.

When *any* number of quantities, either *different*, or *repetitions* of the same, are united together so as to form but one, we term the process, simply, "Addition." When the quantities to be added are the *same*, but we may have *as many* of them as we please, it is called "Multiplication;" when they are not only the *same*, but their number is indicated by *one of them*, the process belongs to "Involution." That is, addition restricts us neither as to the kind, nor the number of the quantities to be added; multiplication restricts us as to the kind, but not the number; involution restricts us both as to the kind and number. All, however, are really comprehended under the same rule—*addition.*

ADDITION.

7. The sum of two or more numbers is a number which contains as many units, and no more, as are found in all the given numbers.

8. Addition is the process of finding the sum of two or more numbers.

9. The quantities to be added together are called *addends*, and the result of the addition is called the *sum* of the addends.

10. Only those quantities can be added which have the same unit, or, in other words, which are of the same denomination.

Thus it is evident that 6 days and 7 miles cannot be added, since the result would neither be 13 days nor 13 miles; nor can 5 shillings and 3 pence be added, as the result would neither be shillings nor pence. Similarly, we cannot add units and tens, or tenths and hundredths, or units and sevenths, &c.

11. Hence, in writing down the addends preparatory to adding, we must be careful to set units of the same denomination in the same vertical column, *i. e.* units under units, tens under tens, hundreds under hundreds, &c.; shillings under shillings, pence under pence, &c.; miles under miles, furlongs under furlongs, rods under rods, &c.

EXERCISE 9.

(1)
Apples.
Addends $\begin{cases} 2 \\ 3 \\ 2 \end{cases}$
Sum of Addends 7

(2)
Shillings.
Addends $\begin{cases} 9 \\ 8 \\ 7 \end{cases}$
Sum of Addends 24

(3)
Addends $\begin{cases} 9 \\ 7 \\ 6 \\ 8 \end{cases}$
Sum of Addends 30

(4) cwt.	(5) pence.	(6) sevenths.	(7) horses.	(8) tens.	(9) millionths.	(10) $	(11) miles.
9	4	6	1	7	6	9	7
6	7	5	9	8	9	8	1
9	8	4	8	9	8	1	2
8	9	3	7	6	3	2	3
7	6	5	4	5	2	3	4
39	34	23	30	35	28	28	17

12. Let it be required to add together 987 and 689.

I.	II.	III.	IV.	V.
987	987	987	987	987
689	689	689	689	689
1500	160	16	16	1676
160	1500	160	16	
16	16	1500	15	
1000	70	6	1676	
600	600	70		
70	6	600		
6	1000	1000		
1676	1676	1676		

EXPLANATION.—We place the given numbers, 987 and 689, under each other, according to (11) and draw a line to separate the addends from the sum.

It is manifest that so long as we add the units of the several orders it is quite immaterial whether we commence at the highest, at the lowest, or at an intermediate denomination.

In the first of the above operations we have commenced continually at the highest or left-hand order. The hundreds added make 15 hundreds or one thousand and five hundred, which we set down; the tens added make 16 tens, equal to 1 hundred and 6 tens, and the units added, make 16 units, equal to 1 ten and 6 units, all of which we set down in their appropriate columns.

Next considering the partial sums 1500, 160, and 16, as so many new addends, we proceed similarly with them and obtain a new set of partial sums, viz.: 1000, 600, 70, and 6. But, from the principles of notation (Sec. I.), these last numbers (*i. e.* 1000, 600, 70, and 6) may be written in one line, thus, 1676, which therefore is the sum of the addends 987 and 689.

In (II), (III), (IV), (V), the same result is obtained by a slightly different process.

In (II) we have commenced at the *tens*, and in (III), (IV), and (V), at the units or lowest order. (IV) is simply (III) with the unnecessary 0's omitted.

(V) is (IV) somewhat modified as follows:—9 *units* and 7 units make 16 *units*, equal to 6 *units*, which we set down, and one *ten* which we carry to the next column or column of tens; 1 ten and 8 tens make 9 tens, and 8 tens make 17 tens, equal to 7 tens, which we set down, and 1 hundred, which we carry to the column of hundreds; 1 hundred and 6 hundreds make 7 hundreds, and 9 hundreds make 16 hundreds, equal to 6 hundreds and 1 thousand, both of which we set down.

13. From (I), (II), and (III), it is manifest that it is *as legitimate* to commence at the lowest denomination as at the highest: and from (IV) and (V), that it is *most convenient* to commence at the lowest denomination.

14. From (V) we learn that when we have obtained the sum of the units, in any column, we reduce it to the next higher denomination, and setting down the remainder under the column added, carry the units of the next higher denomination to their proper column.

15. The reasoning in (12), (13), and (14) applies to any numbers whatever, whether abstract or denominate, and from it, for addition, we deduce the following general—

RULE.

Write down the numbers so that units of the same denomination shall fall in the same column (*Arts.* 10 *and* 11).

Draw a line beneath the addends (*Art.* 12).

Add up the units of the lowest denomination and divide their sum by so many as make one of the denomination next higher (*Arts.* 13 *and* 14).

Set down the remainder and carry the quotient to the next higher denomination (*Art.* 14).

Proceed in the same manner through all the denominations to the last.

EXAMPLE.
698·9649
84·76
9·896
98·462
989·9
1881·9829

16. We commence at the lowest order or tenths of thousandths. There being nothing to add to the 9 tenths of thousandths, we simply set down the 9 in its appropriate column. Next we add the thousandths, thus:—2 thousandths and 6 thousandths are 8 thousandths and 4 thousandths are 12 thousandths, which are equal to 2 thousandths and 1 hundredth. The 2 thousandths we write down in its own column and carry the hundredth to the column of hundredths. Next we add the column of hundredths, thus:—1 hundredth (carried) and 6 hundredths make 7 hundredths and 9 hundredths make 16 hundredths, and 6 hundredths make 22 hundredths and 6 hundredths make 28 hundredths, which are equal to 8 hundredths and two tenths. We set down the 8 hundredths and carry the two tenths to the next column or column of tenths. Addi[illegible]e tenths we find their sum to be 39 tenths, equal to 9 tenths, which we set down, and 3 units which we carry. The simple units added make 41 units, equal to 1 unit, which we set down and 4 tens which we carry; the tens added make 38 tens, equal to 8 tens and 3 hundreds; the hundreds added (with the three hundreds we carry) make 18 hundreds, or 8 hundreds, and 1 thousand, both of which we set down in their proper columns.

EXAMPLE.
$69·89
11·56
73·42
91·89
$246·76

17. We commence as in (16) with the lowest denomination, which, in this example, is cents. 89 cents and 42 cents and 56 cents and 89 cents, added, make 276 cents. But every 100 cents make one dollar, 276 cents are therefore equal to 2 dollars and 76 cents. The 76 cents we set down in their proper place and carry the 2 dollars to the column of dollars.

18. Example.—Add together £52 17s. 3¾d., £47 5s. 6¼d., and £66 14s. 2½d.

£	s.	d.	
52	17	3¾	
47	5	6¼	addends.
66	14	2½	
£166	17	0½	

½ and ¼ make three farthings, which, with ¾, make 6 farthings; these are equivalent to one of the next denomination, or that of pence, to be carried, and two of the present, or one half-penny, to be set down. 1 penny (carried) and 2 are 3, and 6 are 9, and 3 are 12 pence—equal to one of the next denomination, or that of shillings, to be carried, and no pence to be set down; we therefore put a cipher in the pence place of the sum. 1 shilling (carried) and 14 are 15, and 5 are 20, and 17 are 37 shillings—equal to one of the next denomination, or that of pounds, to be carried, and 17 of the present, or that of shillings, to be set down. 1 pound and 6 are 7, and 7 are 14, and 2 are 16 pounds—equal to 6 units of pounds, to be set down, and 1 ten of pounds to be carried; 1 ten and 6 are 7 and 4 are 11 and 5 are 16 tens of pounds, to be set down.

When the addends are very numerous, we may divide them into two or more parts by horizontal lines, and, adding each part separately, may afterwards find the amount of all the sums.

EXAMPLE.

$$\left.\begin{array}{l}\left.\begin{array}{rrr}\text{£} & \text{s.} & \text{d.}\\ 57 & 14 & 2\\ 32 & 16 & 4\\ 19 & 17 & 6\\ 8 & 14 & 2\\ 32 & 5 & 9\\ \hline 47 & 6 & 4\end{array}\right\} = \begin{array}{rrr}\text{£} & \text{s.} & \text{d.}\\ 151 & 7 & 11\end{array}\\ \left.\begin{array}{rrr}32 & 17 & 2\\ 56 & 8 & 9\\ 27 & 4 & 2\\ 52 & 4 & 4\\ 37 & 8 & 2\end{array}\right\} = \begin{array}{rrr}253 & 8 & 11\end{array}\end{array}\right\} = \begin{array}{rrr}\text{£} & \text{s.} & \text{d.}\\ 404 & 11 & 10\end{array}$$

Or, in adding each column, we may put down an asterisk, thus*, as often as we come to a quantity which is at least equal to that number of the denomination added which is required to make one of the next—carrying forward what is above this [illegible], if anything, and putting the last remainder, or—when there is nothing left at the end—a cypher under the column;—we carry to the next column one for every asterisk. Using the same example.

£	s.	d.
57	*14	2
32	16	4
19	*17	*6
8	*14	2
32	5	*9
47	*6	4
32	17	2
56	*8	*9
27	4	2
52	4	4
37	8	2
404	11	10

2 pence and 4 are 6, and 2 are 8, and 9 are 17 pence—equal to 1 shilling and 5 pence; we put down a dot or an asterisk and carry 5. 5 and 2 are 7, and 4 are 11 and 9 are 20 pence—equal to 1 shilling and 8 pence; we put down a dot or an asterisk and carry 8. 8 and 2 are 10 and 6 are 16 pence—equal to 1 shilling and 4 pence; we put down a dot and carry 4. 4 and 4 are 8 and 2 are 10—which being less than 1 shilling, we set down under column of pence to which it belongs, &c. We find on adding them up, that there are three dots; we therefore carry 3 to the column of shillings. 3 shillings and 8 are 11, and 4 are 15, and 4 are 19, and 8 are 22 shillings—equal to 1 pound and 2 shillings: we put down a dot and carry 2. 2 and 17 are 19, &c.

Care is necessary, lest the dots, not being distinctly marked, may be considered as either too few or too many. This method, though now but little used, seems a convenient one.

PROOF OF ADDITION.

19. FIRST METHOD.—*Go through the process again, beginning at the top and adding downwards.*

This method of proof is merely doing the *same* work twice, in a slightly different manner.

SECOND METHOD.—*Separate the addends into two parts. Add each part separately, in the usual way, and then add their sums. If the last sum is the same as that found by the first addition, the work may be presumed to be correct.*

This method of proof is founded on the axiom that "the whole is equal to the sum of all its parts."

Example.—Find the sum of 509267, 235809, 72910, and 83925.

OPERATION.
509267
235809
72910
83925
Sum 901911

PROOF BY SECOND METHOD.

	509267	72910
	235809	83925
Partial sums	745076	156835

First partial sum....745076
Second partial sum 156835
Proof...... 901911

Exercise 10.

(1) Dollars.	(2) Bushels.	(3) Days.	(4) Acres.	(5) Dollars.	(6) Pounds.
15	76	765	392	5832	98764
26	48	381	446	8907	8753
18	59	872	872	4671	76
61	81	315	969	6789	9889
120	264	2333	2679	26199	117482

(7—30)

The sum of the numbers in each row of the following table, whether taken vertically or horizontally, or from corner to corner, is 24156. Let the pupil be required to make these 24 distinct additions.*

TABLE.

2016	4212	1656	3852	1296	3492	936	3132	576	2772	216
252	2052	4248	1692	3888	1332	3528	972	3168	612	2412
2448	288	2088	4284	1728	3924	1368	3564	1008	2808	648
684	2484	324	2124	4320	1764	3960	1404	3204	1044	2844
2880	720	2520	360	2160	4356	1800	3600	1440	3240	1080
1116	2916	756	2556	396	2196	3996	1836	3636	1476	3276
3312	1152	2952	792	2592	36	2232	4032	1872	3672	1512
1548	3348	1188	2988	432	2628	72	2268	4068	1908	3708
3744	1584	3384	828	3024	468	2664	108	2304	4104	1944
1980	3780	1224	3420	864	3060	504	2700	144	2340	4140
4176	1620	3816	1260	3456	900	3096	540	2736	180	2376

---

* This table is formed by multiplying the numbers in the magic square of 11 by 36.

l 83925.

D.

)
nds.
764
753
76
889

482

e, whether
is 24156.
ns.*

216
2412
648
2844
1080
3276
1512
3708
1944
4140
2376

re of 11 by 36.

---

(31)	(32)	(33)	(34)	(35)	(36)
74564	5676	76746	67674	42·37	0·87
7674	1567	71207	75670	56·84	5·273
376	63	100	36	27·92	8·127
6	6767	56	77	62·41	25·63
82620					

(37)	(38)	(39)	(40)
3·785	85·742	0·00007	5471·3
20·766	6034·82	0·06236	563·47
0·253	57·8563	0·572	21·502
10·004	712·52	0·21	0·0007
34·808			

(41)	(42)	(43)	(44)
81·0235	0·0007	8456·5	576·34
576·03	5000·0	0·37	4000·005
4712·5	427·0	8456·302	213·5
6·53712	37·12	0·007	2753·0
5376·09062			

MONEY.

(45)			(46)			(47)			(48)		
£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
4567	14	6¼	76	14	7	3767	13	11	5674	17	6½
776	15	7½	667	13	6	4678	14	10	4767	16	11½
76	17	9¾	67	15	7	767	12	9	3466	17	10¾
51	0	10¼	5	4	2	10	11	5	5984	2	2¼
44	5	6		3	4	3	4	11	8762	9	9
5516	14	3¾									

AVOIRDUPOIS WEIGHT.

(49)			(50)			(51)			(52)		
cwt.	qrs.	lb.	cwt.	qrs.	lb.	cwt.	qrs.	lb.	cwt.	qrs.	lb.
76	3	14	476	1	24½	447	1	7	14	2	12
37	2	15	756	3	21¼	576	1	6	3	3	7
14	1	11	767	1	16	467	1	7½		2	15
128	3	15	567	2	15	563	1	6	7	0	3
			973	1	12	428	0	0¼			14

TROY WEIGHT.

(53)

lb.	oz.	dwt.	grs.
7	0	5	9
5	6	6	7
9	5	6	8
21	11	18	0

(54)

lb.	oz.	dwt.	grs.
57	9	12	14
67	9	11	11
66	8	10	5
74	6	5	3
12	3	5	4

(55)

lb.	oz.	dwt.	grs.
87	3	7	12
	11	12	3
		16	14
44	12	10	13
67	8	9	10

TIME.

(56)

yrs.	ds.	hrs.	ms.
99	359	9	56
88	0	8	57
77	120	7	49
265	115	2	42

(57)

yrs.	ds.	hrs.	ms.
60	90	0	50
6	76	1	57
		3	58
6	1	2	0

(58)

yrs.	ds.	hrs.	ms.
50	127	7	50
	120	9	44
76	121	11	44
6	47	3	41
8	9	11	17

CLOTH MEASURE.

(59)

yds.	qrs.	nls.
567	3	2
476	1	0
72	3	3
5	2	1
1122	2	2

(60)

yds.	qrs.	nls.
147	3	3
173	1	0
148	2	1
92	3	2

(61)

yds.	qrs.	nls.
157	2	1
143	3	2
	1	2
54	0	3

(62)

yds.	qrs.	nls.
156	1	1
176	3	1
54	1	0
573	2	3

CANADIAN MONEY.

(63)	(64)	(65)	(66)
$978·63	$69·42	$719·43	$9863·47
492·29	189·87	912·99	986·10
83·43	674·29	68·68	91·89
729·47	86·43	50·00	7·45
9·00	982·78	9·73	·98
$2292·82	$	$	$

67. 0·4+74·47+37·007+75·05+747·077=934·004.
68. 56·05+4·75+0·007+36·14+4·672=101·619.
69. 0·76+0·0076+76+0·5+5+0·05=82·3176.
70. 0·5+0·005+5+50+500=555·505.
71. 0·367+56·7+762+97·6+471=1387·667.

72. Add eight hundred and fifty-six thousand, nine hundred and thirty-three; one million, nine hundred and seventy-six thousand, eight hundred and fifty-nine; two hundred and three millions, eight hundred and ninety-five thousand, seven hundred and fifty-two. *Ans.* 206729544.

73. Add three millions, and seventy-one thousand; four millions, and eighty-six thousand; two millions, and fifty-one thousand; one million; twenty-five millions, and six; seventeen millions, and one; ten millions, and two; twelve millions, and twenty-three; four hundred and seventy-two thousand, nine hundred and twenty-three; one hundred and forty-three thousand; one hundred and forty-three millions. *Ans.* 217823955.

74. Add one hundred and thirty-three thousand; seven hundred and seventy thousand; thirty-seven thousand; eight hundred and forty-seven thousand; thirty-three thousand; eight hundred and seventy-six thousand; four hundred and ninety-one thousand. *Ans.* 3187000.

75. Add together one hundred and sixty-seven thousand; three hundred and sixty-seven thousand; nine hundred and six thousand; two hundred and forty-seven thousand; ten thousand; seven hundred thousand; nine hundred and seventy-six thousand; one hundred and ninety-five thousand; ninety-seven thousand. *Ans.* 3665000.

APPLICATIONS.

1. How many miles is it from the lower end of Lake Huron to the Gulf of St. Lawrence, passing through the River St. Clair, 25 miles long; Lake St. Clair, 20 miles; River Detroit, 23 miles; Lake Erie, 250 miles; Niagara River, 34 miles; Lake Ontario, 180 miles; and the River St. Lawrence, 750 miles long? *Ans.* 1282 miles.

2. The city of Toronto has a population of about 50000; Hamilton, 25000; Kingston, 15000; London, 10000; Ottawa, 10000; Montreal, 75000; and Quebec, 45000. What is the population of these seven cities taken together? *Ans.* 230000.

3. In the year 1856 Canada exported:—Produce of the mine, $165000; produce of the sea, $500000; produce of the forest, $10000000; animals and their produce, $2500000; agricultural products, $15000000; manufactures and ships, $1600000; and various other products to the amount of $2235000. What was the total value of Canadian exports for that year? *Ans.* $32000000.

4. A wholesale merchant sells, during the year, goods to the amount of $11080 in Toronto; $9427 in Galt; $1798 in Berlin;

$16423 in Hamilton; $7496 in Guelph; $6429 in Woodstock; $5297 in Chatham; and $8426 in Goderich. Required the amount of the year's sales. *Ans.* $66376.

5. The Grand Trunk Railway is 962 miles long, and cost $60000000; the Great Western is 229 miles long, and cost $14000000; the Ontario, Simcoe, and Huron is 95 miles long, and cost $3300000; the Toronto and Hamilton is 38 miles long, and cost $2000000. What is the aggregate length and cost of these four roads?
Ans. Length, 1324 miles, and cost $79300000.

6. The circulation of promissory notes for the four weeks ending February 3, 1844, was as follows:—Bank of England, about £21228000; private banks of England and Wales, £4980000; Joint Stock Banks of England and Wales, £3446000; all the banks of Scotland, £2791000; Bank of Ireland, £3581000; all the other banks of Ireland, £2429000; what was the total circulation?
Ans. £38455000.

7. Chronologers have stated that the creation of the world occurred 4004 years before Christ; the deluge, 2348; the call of Abraham, 1921; the departure of the Israelites from Egypt, 1491; the foundation of Solomon's temple, 1012; the end of the captivity, 536. This being the year 1859, how long is it since each of these events?

Ans. From the creation, 5863 years; from the deluge, 4207; from the call of Abraham, 3780; from the departure of the Israelites, 3350; from the foundation of the temple, 2871; and from the end of the captivity, 2395.

8. Add together the following:—2d., about the value of the Roman sestertius; 7½d., that of the denarius; 1½d., a Greek obolus; 9d., a drachma; £3 15s., a mina; £225, a talent; 1s. 7d., the Jewish shekel; and £342 3s. 9d., the Jewish talent. *Ans.* £571 2s.

9. Add together 2 dwt. 16 grains, the Greek drachma; 1 lb. 1 oz. 1 dwt., the mina: 67 lb. 7 oz. 5 dwt., the talent.
Ans. 68 lb. 8oz. 8dwt. 16 grains.

10. What was the population of the British provinces in North America in 1834, the population of Lower Canada being stated at 549005, of Upper Canada, 336461; of New Brunswick, 152156; of Nova Scotia and Cape Breton, 142548; of Prince Edward's Island, 32292; of Newfoundland, 75,000? *Ans.* 1287462.

11. A owes to B £567 16s. 7½d.; to C £47 16s.; and to D £56 0s. 1d. How much does he owe in all? *Ans.* £671 12s. 8½d.

12. A man has owing to him the following sums:—£3 10s. 7d.; £46 0s. 7½d.; and £52 14s. 6d. How much is the entire?
Ans. £102 5s. 8½d.

13. A merchant sends off the following quantities of butter:—47 cwt. 2 qrs. 7 lb.; 38 cwt. 3 qrs. 8 lb.; and 16 cwt. 2 qrs. 20 lb. How much did he send off in all? *Ans.* 103 cwt. 10 lb.

14. A merchant receives the following quantities of tallow, viz.:—

13 cwt. 1 qr. 6 lb.; 10 cwt. 3 qrs. 10 lb.; and 9 cwt. 1 qr. 15 lb. How much has he received in all? *Ans.* 33 cwt. 2 qrs. 6 lb.

15. A silversmith has 7 lb. 8 oz. 16 dwts.; 9 lb. 7 oz. 3 dwts.; and 4 lb. 1 dwt. What quantity has he? *Ans.* 21 lb. 4 oz.

16. A merchant sells to A, 76 yards 3 quarters 2 nails; to B, 90 yards 3 quarters 3 nails; and to C, 190 yards 1 nail. How much has he sold in all? *Ans.* 357 yards 3 quarters 2 nails.

17. A merchant in Toronto sells goods to the following amounts during the week, viz.:—Monday, $429·38; Tuesday, $711·43; Wednesday, $419·87; Thursday, $1080·42; Friday, $1304·65; Saturday, $2498·91. Required the whole amount of the week's sales. *Ans.* $6444.66.

18. Looking over my last month's expenditure, I find that I have paid the following sums, viz.:—Baker's bill, $5·73; Butcher's bill, $20·91; Groceries, $12·75; Fruit, $3·29; Rent, $16·25; Servants' wages, $10; Tailor's account, $17·87; Shoemaker's bill, $11·63; and sundries, $9·47. Required how much I paid in all. *Ans.* $107·90.

19. Add together $607·19; $298·97; $789·37; $1723·10; and $123·00. *Ans.* $3542·13.

20. A farmer sells seven loads of wheat, the first containing 1763 lbs., the second 1827 lbs., the third 1329 lbs., the fourth 1901 lbs., the fifth 1666 lbs., the sixth 1879 lbs., and the seventh 1185 lbs. What was the aggregate weight of the seven loads, and how many bushels did they contain? *Ans.* 11550 lbs. or 192½ bushels.

Note.—The bushels are found by dividing the aggregate weight by 60 lbs., the weight of one bushel.

21. Having effected an insurance on my household furniture, &c., I am required to make a detailed statement of its value. I find this to be as follows:—Carpets, $250·00, table and bed linen, $90·88, beds and bedding, $173·60, furniture, $791·23, pictures and engravings, $207·18, books, $1649·19, plate and plated ware, $307·18. Required the total value of my household furniture. *Ans.* $3469·26.

22. Toronto has a population of 45000, Hamilton, 20000, Brockville, 4000, Prescott, 2500, Kingston, 15000, Ottawa City, 10000, Chatham, 4000, Goderich, 2000, London, 10000, Port Hope, 4000, Cobourg, 5000, Montreal, 70000, and Quebec, 50000. What is the entire population of these 13 cities and towns? *Ans.* 241500.

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20. The pupil should not be allowed to leave addition until he can *read* up the column without hesitation. For instance, in the following questions, which are inserted for the sake of practice in rapid addition, he should not be permitted to *spell* the columns thus, 6 and 4 are 10, and 4 are 14, and 4 are 18, and 5 are 23, &c., but should be required to *read* them, *i. e.*, simply touch each digit with his pencil and name the sum, thus:—6, 10, 14, 18, 23, 31, 32, 35, 42, 43, 44, 49, 53, &c., &c.

I.	II.	III.	IV.
244658	275634	135790	123456
492327	386731	246824	786123
635425	987654	135790	456789
321465	321456	864212	123456
732849	989123	579246	788123
376731	456789	835792	459789
935746	123456	468357	123456
847963	789123	924689	789123
745143	456789	753246	456789
234561	123456	835792	123456
746874	789123	468357	789123
934746	456789	924683	456789
872345	123459	579246	123456
934756	789123	835798	789123
842345	456789	642875	456789
873456	123456	334683	123456
864580	789123	579864	789123
234672	456789	297531	456789
325871	246842	135795	871178
479234	357931	246834	936639
845645	642248	824248	248842
823456	756139	357964	525255
245734	246842	872278	736376
872475	657931	375946	875578
896731	642248	624862	473468
456841	753139	375937	934579
314567	246842	872459	894645
814563	357931	837645	123875
427831	642248	644875	767457
932768	753913	472963	875345
456345	375913	875847	874563
345634	426428	864314	375534
734734	573931	734561	937565
734564	624824	273475	875734
834756	735813	845675	698945

RECAPITULATION

I. Addition is the process of finding the sum of two or more numbers.

II. The numbers to be added are called *Addends*.

III. The *result* of the addition is called the *sum* of the addends.

IV. In writing numbers down preparatory to adding them, we write units under units, tens under tens, &c., because it is more convenient, since only *like* quantities, *i. e.*, quantities of the same name, can be added together.

V. We draw a line under the addends in order to separate them from the *sum*.

VI. We begin the addition at the column containing the lowest denomination, and work from right to left, because, by so doing, we are enabled to *carry*, from the column added, the number of units of the next higher denomination it contains, to their appropriate column, and thus perform the work by one addition, which would otherwise require two or more.

VII. We divide the sum of the units of any one denomination by the number required to make one of the next higher, in order to know how many we are to carry to the next higher.

VIII. The addition of simple numbers was formerly called Simple Addition; and the addition of compound or denominate numbers, Compound Addition. As the same rule applies to the addition of all numbers, there is no reason why, in a second course, we should treat of the addition of simple and denominate numbers separately.

QUESTIONS.

NOTE.—*Arabic numerals, thus* (14), *refer to the articles of the Section, and Roman numerals, thus* (VI.), *to the Recapitulation.*

1. Into what parts may Arithmetic be divided? (1)
2. Of what does the Arithmetic of whole numbers treat? (1)
3. What rules are included in the Arithmetic of Whole Numbers? (2)
4. Of what does the Arithmetic of Fractions treat? (1)
5. How is the Arithmetic of Fractions divided? (3)
6. How is the unit divided in Vulgar or Common Fractions? (3)
7. How is the unit divided in Decimal Fractions? (3)
8. Of what does the Arithmetic of Ratios treat? (1)
9. What rules of Arithmetic are embraced in the Arithmetic of Ratios? (4)
10. What are the fundamental rules of Arithmetic? (5)
11. Why are they so called? (5)
12. Upon what rules do all the operations of Arithmetic ultimately depend? (6)
13. What is the *sum* of two numbers? (7)
14. What is Addition? (8 or I.)
15. What are addends? (9 or II.)
16. What kind of quantities only can be added? (10)
17. What is the rule for Addition? (15)
18. Why must we place units of the same denomination in the same vertical column? (IV.)

19. Why do we draw a line under the addends? (V.)
20. Why do we begin to add at the lowest denominations? (VI.)
21. Why do we divide the sum of the units of any one denomination by as many as make one of the next higher? (VII.)
22. How do we prove addition? (19)
23. Upon what axiom is the 2nd method of proof founded? (19)
24. So far as the result is concerned, does it make any difference where we commence to add? (12)
25. Exhibit the work when we commence adding at the left-hand side, or highest denomination. (12)
26. When the addends are very numerous, what plans may we adopt? (18)
27. Upon what principle does the former of these plans proceed? (19)
28. What different rules were formerly made in addition? (VIII.)
29. Is this distinction necessary? Why not? (VIII.)
30. Illustrate the difference between *spelling* and *reading* in addition. (20)

SUBTRACTION.

21. Subtraction is the process of finding the difference between two numbers.

22. The greater of the two given numbers, or that which is *to be lessened*, is called the *Minuend* (Lat. *Minuendus*, "*to be lessened*"); the smaller, or that which is *to be Subtracted*, the *Subtrahend* (Lat. *Subtrahendus*, "to be subtracted").

23. If anything is left after making the subtraction, it is called the *remainder*, *difference*, or *excess*.

24. Only quantities of the same denomination (i. e. which have the same unit) can be subtracted the one from the other.

25. Subtraction is indicated by $-$, called the minus or negative sign. Thus $5-4=1$, read five minus four equal to one, indicates that if 4 is subtracted from 5, unity is left.

Quantities connected by the negative sign cannot be taken, indifferently, in any order; because, for example, $5-4$ is not the same as $4-5$. In the former case the positive quantity is the greater, and 1 (which means $+1$) is left; in the latter, the negative quantity is the greater, and -1, or one to be subtracted, still remains. To illustrate yet further the use and nature of the signs, let us suppose that we *have* five pounds and *owe* four;—the five pounds we *have* will be represented by 5, and our debt by -4; taking the 4 from the 5, we shall have 1 pound ($+1$) remaining. Next, let us suppose that we *have* only four pounds and *owe* five; if we take the 5 from the 4 (that is, if we pay as far as we can) a debt of one pound, represented by -1, will still remain; consequently $5-4=1$; but $4-5=-1$.

26. When several numbers, connected by the signs + and − are placed within brackets, thus, $(7+4-6-3+9,)$ the whole expression is to be considered as one quantity. The negative sign before such an expression indicates that the *value* of the whole expression within the brackets, is to be subtracted, or, what amounts to the same thing, that the numbers having the sign + before them are to be subtracted, and those having the sign −, added. Hence a minus sign before a bracket, has the effect of changing the signs of all the quantities within the brackets, when the brackets are removed. So, also, when we desire to place a quantity within brackets, we must change its sign, if the sign preceding the first bracket be minus.

The following examples will show how the brackets affect numbers, according as we make them include an additive, or a subtractive quantity:—

$27-4+7-3=27$
$27-(4+7-3)=19$
But $27-(4-7+3)=27$. [changing all the signs of the original quantities, but the first.]
Again $48+7-3-8+7-2=49$.
$48+(7-3-8+7-2)=49$; what is *in* the brackets being additive, it is not necessary to change any signs.
$48+7-(3+8-7+2)=49$; it is now necessary to change all the signs *in* the brackets.
$48+7-3-(8-7+2)=49$; it is necessary in this case, also, to change the signs.
$48+7-3-8+(7-2)=49$; it is not necessary in this case.

27. When the numbers are small they can be subtracted mentally, thus: from 6 shillings take 4 shillings, and the result is evidently 2 shillings; from 9 pounds take 4 pounds, and the remainder is 5 pounds; from 16 days, take 9 days, and the remainder is 7 days; from 14 sixteenths take 5 sixteenths, and the remainder is 9 sixteenths, &c.

When the numbers are too large to be conveniently retained in the mind, they may be written as in addition.

EXAMPLE 1.—From 97 take 43, that is, from 9 tens and 7 units take 4 tens and 3 units.

OPERATION.
90+7 or 97=Minuend.
40+3 or 43=Subtrahend.
50+4 or 54=Remainder.

EXPLANATION.—3 units from 7 units leaves 4 units, and 40 units or 4 tens from 90 units or 9 tens, leave 50 units or 5 tens.

EXAMPLE 2.—Let it be required to subtract 746 from 978, or from 900+70+8 to take 700+40+6

OPERATION.

	hundreds.	tens.	units
900+70+8 or	9	7	8
700+40+6 or	7	4	6
200+30+2 or	2	3	2

EXPLANATION.—6 units from 8 units, and 2 units remain; 40 units or 4 tens from 70 units or 7 tens, and 30 units or 3 tens remain; and 700 units or 7 hundreds, from 900 units or 9 hundreds, and 200 units, or 2 hundreds remain.

EXAMPLE 3.—From 842 take 661.

EXPLANATION.—In placing the subtrahend under the minuend, in this example, we find that, while we can subtract the units from the units, we cannot subtract the tens from the tens, since we have 6 tens in the subtrahend and only 4 tens in the minuend. We get over this difficulty by considering the minuend to be, not 800+40+2, but 700+140+2, or in other words, we *borrow* one of the order of hundreds and reduce it to tens. Now we have 1 unit from 2 units and 1 unit remains; 60 units or 6 tens from 140 units or 14 tens, and 80 units or 8 tens remain; 600 units or 6 hundreds, from 700 units or 7 hundreds, and 100 units or 1 hundred remain.

OPERATION.

$$\begin{array}{lcrcr}
\text{I.} & & \text{II.} & & \text{III.} \\
842 & \text{or} & 800+40+2 & \text{or} & 700+140+2 \\
661 & \text{or} & 600+60+1 & \text{or} & 600+\ \ 60+1 \\
\hline
181 & \text{or} & & & 100+80+1
\end{array}$$

EXAMPLE 4.—Let it be required to subtract 3 cwt. 2 qrs. 7 lbs. from 9 cwt. 1 qr. 8 lbs.

EXPLANATION.—As we cannot subtract 2 qrs. from 1 qr. we *borrow* 1 cwt. and reduce it to quarters. The 9 cwt. 1 qr. 8 lb. we then consider as 8 cwt. 5 qrs. 8 lb. and from it subtract the 3 cwt. 2 qrs. 7 lb. Thus, 7 lbs. from 8 lbs. and 1 lb. remains; 2 qrs. from 5 qrs. and 3 qrs. remain: and 3 cwt. from 8 cwt. and 5 cwt. remain.

OPERATION.

$$\begin{array}{ccccccc}
\text{cwt.} & \text{qrs.} & \text{lb.} & & \text{cwt.} & \text{qrs.} & \text{lb.} \\
9 & 1 & 8 & = & 8 & 5 & 8 \\
3 & 2 & 7 & = & 3 & 2 & 7 \\
\hline
5 & 3 & 1 & & 5 & 3 & 1
\end{array}$$

28. Hence, to find the difference between two numbers, we deduce the following:—

RULE.

Write the subtrahend under the minuend, so that units of the same denomination may be in the same vertical column (24). *Draw a line under the subtrahend to separate it from the remainder. Subtract each digit in the subtrahend from the one over it in the minuend, beginning at the lowest denomination.*

When the units of any one denomination of the minuend fall short of those of the same denomination in the subtrahend, borrow one of the next higher denomination in the minuend, reduce it to its equivalent units of the required denomination, add them to the units of that denomination given in the minuend, and from their sum subtract the units of that denomination given in the subtrahend.

29. The following is the complete work of a question in Subtraction:

EXAMPLE 5.—From 6400 lbs. 0 oz. 0 dwt. 7·0006 grs. take 987 lbs. 3 oz. 17 dwt. 22·6349 grs.

OPERATION.

$$\begin{array}{llllll}
(10)\ 9\ \ 9 & 11 & 19 & 24\cdot\ 9\ \ 9\ \ 9 & & \\
5\ \ 3\ \cancel{10}\ \cancel{10} & \cancel{12} & \cancel{20} & 6\cdot\cancel{10}\cancel{10}\cancel{10}(10) & & \\
\cancel{6}\ \cancel{4}\ \ 0\ \ 0\ \text{lbs.} & 0\ \text{oz.} & 0\ \text{dwt.} & \cancel{7}\cdot0\ \ 0\ \ 0\ \ 6 & \text{grs.} & \text{Minuend.} \\
\ \ \ 9\ \ 8\ \ 7 & 3 & 17 & 22\cdot6\ \ 3\ \ 4\ \ 9 & & \text{Subtrahend.} \\
\hline
5\ 4\ 1\ 2 & 8 & 2 & 8\cdot3\ \ 6\ \ 5\ \ 7 & & \text{Remainder.}
\end{array}$$

EXPLANATION.—Here, as we cannot take 9 tenths of thousandths of a grain from 6 tenths of thousandths of a grain, we borrow one grain, there being no tenths, hundredths, or thousandths in the minuend. Now this one grain is equivalent to ten of the order of tenths of grains. Borrow one tenth and there remain 9 tenths, and the one tenth we borrowed is equal to 10 hundredths. Borrow 1 hundredth, there remain 9 hundredths, and the one hundredth we borrowed is equal to 10 thousandths. Borrow 1 thousandth, there remain 9, and the 1 thousandth is equal to 10 of the order of tenths of thousandths—the order for which it was necessary to borrow. 10 of the order of tenths of thousandths of grains and 6 of the order of tenths of thousandths of grains, make 16, from which take 9 of the order of tenths of thousandths of grains, and there remain 7 of the order of tenths of thousandths of grains; 4 of the order of thousandths from 9 of the order of thousandths and 5 of the order of thousandths remain; 3 of the order of hundredths from 9 of the order of hundredths and 6 hundredths remain; 6 tenths from 9 tenths and 3 tenths remain.

Again, as we cannot take 22 grains from 6 grains, we borrow from the next available higher order, which, in this case, is hundreds of pounds. 1 of the order of hundreds of pounds reduced, as above, to its equivalent lower denomination, is equal to 9 tens of lbs., 9 units of lbs. 11 oz. 19 dwt. 24 grs. 24 grains, added to 6, make 30 grains, and 22 grains from 30 grains, leave 8 grains; 17 dwt. from 19 dwt. leave 2 dwt.; 3 oz. from 11 oz. leave 8 oz.; 7 units of lbs. from 9 units of lbs. leave 2 units of lbs.; 8 tens of lbs. from 9 tens of lbs. leave 1 ten of lbs. We cannot take 9 hundreds of lbs. from 3 hundreds of lbs., so we are compelled to borrow 1 of the order of thousands of lbs., which is equal to 10 hundreds of lbs., and 3 hundreds of lbs., make 13 hundreds of lbs.; 9 hundreds of lbs. from 13 hundreds of lbs. and 4 hundreds of lbs. remain; 0 thousands of lbs. from 5 thousands of lbs. and 5 thousands of lbs. remain.

30. If any digit of the minuend be smaller than the corresponding digit of the subtrahend, practically, we can proceed in either of two ways. First, we may increase that denomination of the minuend which is too small, by borrowing *one* from the next higher (considered as so many of the lower denomination, or that which is to be increased), and adding it to those of the lower, already in the minuend. In this case we alter the form, but not the value of the minuend; which, in the example given below, would become—

hundreds.	tens.	units.	
7	8	12	= 792, the minuend.
4	2	7	= 427, the subtrahend.
3	6	5	= 365, the difference.

Or, secondly, we may add equal quantities to both minuend and subtrahend, which will not alter the difference; then we would have—

hundreds.	tens.	units.	
7	9	2+10	= 792+10, the minuend +10.
4	2+1	7	= 427+10, the subtrahend +10.
3	6	5	= 365+ 0, the same difference.

In this mode of proceeding we do not use the *given* minuend and subtrahend, but others which produce the same remainder.

PROOF OF SUBTRACTION.

31. FIRST METHOD.—*Add together the remainder and subtrahend; the sum should be equal to the minuend.*

For the remainder expresses by how much the subtrahend is smaller than the minuend: adding, therefore, the remainder to the subtrahend, should make it equal to the minuend; thus,

	8754	minuend.
	5839	subtrahend.
	2915	difference.
Sum of difference and subtrahend,	8754	= minuend.

Second Method.—*Subtract the remainder from the minuend, and what is left should be equal to the subtrahend.*

For the remainder is the excess of the minuend over the subtrahend; therefore, taking away this excess should leave both equal; thus,

8634 minuend.
7985 subtrahend.
649 remainder.

Proof: 8634 minuend.
649 remainder.
New remainder, 7985 = subtrahend.

In practice, it is sufficient to set down the quantities once; thus,

8634 minuend.
7985 subtrahend.
649 remainder.
Difference between remainder and minuend, 7985 = subtrahend.

Exercise 11.

	(1)	(2)	(3)	(4)	(5)
From	11000000	3000001	8000800	8000000	4040053
Take	9919919	2199077	377776	62358	220202
	1080081				

	(6)	(7)	(8)	(9)	(10)
From	85·73	864·5	594·763	47·630	52·137
Take	42·16	73·2	85·6	0·078	20·005
	43·57				

	(11)	(12)	(13)	(14)	(15)
From	0·00063	874·32	57·004	47632·0	400·3270
Take	0·00048	5·63705	2·3	0·845003	0·006
	0·00015				

16. 7465676 − 567456 = 6898220.
17. 566789 − 75674 = 491115.
18. 941000 − 5007 = 935993.
19. 97001 − 50077 = 46924.
20. 76734 − 977 = 75757.
21. 56400 − 100 = 56300.
22. 700000 − 99 = 699901.
23. 5700 − 500 = 5200.
24. 9777 − 89 = 9688.
25. 76000 − 1 = 75999.
26. 90017 − 3 = 90014.
27. 97777 − 4 = 97773.
28. 60000 − 1 = 59999.
29. 75477 − 76 = 75401.
30. 7·97 − 1·05 = 6·92.
31. 1·75 − 0·074 = 1·676.
32. 97·07 − 4·769 = 92·301.
33. 7·05 − 4·776 = 2·274.
34. 10·761 − 9·001 = 1·76.
35. ·10009 − 7·121 = 4·97909.
36. 176·1 − 0·007 = 176·093.
37. 15·06 − 7·863 = 7·179.

MONEY.

	(38)	(39)	(40)	(41)
From	$9876·43	$427·63	$721·73	$16·25
Take	987·49	197·21	91·00	9·75
	$8888·94	$230·42	$	$

	(42)	(43)	(44)	(45)
From	$1234·50	$671·98	$286·29	$7.19
Take	999·96	99·67	611·89	1·86
	$234·54	$572·31	$	$

	(46)			(47)			(48)			(49)			(50)		
	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
From	1098	12	6	767	14	8	76	15	6	47	16	7	97	14	6
Take	434	15	8	486	13	9		14	5	39	17	4	6	15	7
	£663	16	10												

	(51)			(52)			(53)			(54)			(55)		
	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
From	98	14	2	47	14	6	97	16	6	147	14	4	560	15	6
Take	77	15	3	38	19	9	88	17	7	120	10	8	477	17	7

AVOIRDUPOIS WEIGHT.

	(56)			(57)			(58)			(59)		
	cwt.	qrs.	lb.	cwt.	qrs.	lb.	cwt.	qrs.	lb.	cwt.	qrs.	lb.
From	200	2	24	175	2	15	9664	2	23	554	0	0
Take	99	3	15	27	2	7	9073	0	24	476	3	5
	100	3	9									

TROY WEIGHT.

	(60)				(61)				(62)			
	lb.	oz.	dwt.	grs.	lb.	oz.	dwt.	grs.	lb.	oz.	dwt.	grs.
From	554	9	19	4	946	0	10	0	917	0	14	9
Take	97	0	16	15			17	23	798	0	18	17
	457	9	2	13								

TIME.

	(63)				(64)				(65)			
	yrs.	ds.	hrs.	ms.	yrs.	ds.	hrs.	ms.	yrs.	ds.	hrs.	ms.
From	767	131	6	30	475	14	13	16	567	126	14	12
Take	476	110	14	13	160	16	13	17	400	0	15	0
	291	20	16	17								

APPLICATIONS.

1. A shopkeeper bought a piece of cloth containing 42 yards for £22 10s., of which he sells 27 yards for £15 15s.; how many yards has he left, and what have they cost him? *Ans.* 15 yards; and they cost him £6 15s.

2. A merchant bought 234 tons, 17 cwt., 1 quarter, 23 lb., and sold 147 tons, 18 cwt., 2 quarters, 24 lb.; how much remained unsold? *Ans.* 86 tons, 18 cwt. 2 qrs. 24 lb.

3. In 1856, the revenue of Canada was as follows:—customs, $4500000; public works, $500000; crown lands, $500000; and casual, $320000. For the same year the expenditure was as follows:—interest on public debt, &c., $1000000; civil government, $225000; legislation, $450000; administration of justice, $450000; education, $380000; collection of revenue, $940000; public works, &c., $1755000. How much did the total revenue of that year exceed the total expenditure? *Ans.* $620000.

4. The census of 1852 gives the population of Upper Canada as 962004, and that of Lower Canada as 890261. By how much did the population of the former exceed that of the latter? *Ans.* 71743.

5. Upper Canada contains 147832 square miles; Lower Canada, 209990 square miles; Nova Scotia and Cape Breton, 18746 square miles; New Brunswick, 27620 square miles; Prince Edward's Island, 2173 square miles; Newfoundland, 36000 square miles; and Hudson's Bay Territory, 2436000 square miles. By how much does the aggregate extent of these British North American Provinces fall short of the total area of the United States—the latter being 2936116 square miles? *Ans.* 57755 square miles.

6. A merchant has 209 casks of butter, weighing 400 cwt. 2 qrs. 14 lb.; and ships off 173 casks, weighing 213 cwt. 2 qrs. 24 lb. How many casks has he left; and what is their weight? *Ans.* 36 casks, weighing 186 cwt. 3 qrs. 15 lb.

7. If from a piece of cloth containing 496 yards, 3 quarters, and 3 nails, I cut 247 yards, 2 qrs., 2 nails, what is the length of the remainder. *Ans.* 249 yards, 1 quarter, 1 nail.

8. A field contains 769 acres, 3 roods, and 20 perches, of which 576 acres, 2 roods, 23 perches, are tilled; how much remains untilled? *Ans.* 193 acres, 37 perches.

9. I owed my friend a bill of £76 16s. 9½d., out of which I paid £59 17s. 10¾.; how much remained due? *Ans.* £16 18s. 10¾d.

10. The population of London is 2363141, and that of Paris is 1053262. How much does the population of London exceed that of Paris? *Ans.* 1309879.

11. The population of Liverpool is 384265, and that of New York 515547. How much does the population of New York exceed that of Liverpool? *Ans.* 131282.

12. Lake Huron contains 20000 square miles: by how much does it exceed the area of Lakes Erie and Ontario—the former containing 11000 square miles, and the latter 7000 square miles? *Ans.* 2000 square miles.

13. A merchant has $6947·87 in bank; $4789·63 in stock; $9491·11 in property; and $14167·93 on his books against his customers: his debts amount to $19478·25. How much is he worth after paying what he owes? *Ans.* $15918·29.

14. What is the value of 6—3+15—4? *Ans.* 14.
15 Of 43+(7—3—14)? *Ans.* 33.
16. Of 47·6—(2+1—24+16—0·34? *Ans.* 52·94.
17. What is the difference between 15+13—6—81 and 15+13—(6—81+62)? *Ans.* 100.

32. Before the pupil leaves subtraction he should be able to take any of the nine digits, continually, from a given number, without stopping or hesitating, thus, in subtracting 7 continually from 94, he should say, 94, 87, 80, 73, 66, 59, &c. In the following examples, which are inserted for practice, he should not be allowed to spell the subtraction, thus, 6 from 9 and 3 remain, 4 from 2, we can't, but 4 from 12 and 8 remain, &c.; but should be required to read as follows:—6, 9..3; 4, 12..8; 9, 13..4; 10, 11..1; 10, 18..8, &c.

(18)
98000460430191816978000041081329
191347813191681473199916199846

(19)
74321913047123098706540456007139
1342345678912345678912345678912

RECAPITULATION.

I. Subtraction is the process of finding the difference between two numbers.

II. The greater of the two numbers is called the *minuend.*

III. The smaller of the two numbers is called the *subtrahend.*

IV. What is left after making the subtraction is called the *remainder* or difference.

V. Only quantities of the same denomination can be subtracted.

VI. Subtraction is indicated by the sign —, which is called minus, or the negative sign.

VII. When several numbers are inclosed in brackets, they are to be considered as constituting only one quantity.

VIII. When a negative sign precedes the first bracket it indicates that all the quantities within the brackets are to have their signs changed when the brackets are removed.

IX. When quantities are removed *into* brackets, preceded by the negative sign, all their signs must be changed.

X. We begin subtraction at the lowest denomination, because it is sometimes necessary to borrow from the higher denominations and reduce.

XI. Instead of thus borrowing and reducing, we may consider any denomination in the minuend increased by as many units of that denomination as make one of the next higher, and then add one to the next higher denomination in the subtrahend. This is merely adding the same quantity under different forms to both minuend and subtrahend, and consequently cannot affect the value of the remainder. (30.)

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*Numbers in Roman numerals, thus* (V), *refer to the Recapitulation; those in Arabic numerals, thus* (25), *refer to the articles of the Section.*

1. What is Subtraction? (I.)
2. What is the minuend? (II.)
3. What is the derivation of the word *minuend?* (22)
4. What is the subtrahend? (III.)
5. What is the derivation of the word *subtrahend?* (22)
6. What is the remainder? (IV.)
7. What kind of quantities can be subtracted? (V.)
8. How is subtraction indicated? (VI.)
9. When several numbers are inclosed together in brackets, how are they to be taken? (VII and 26.)
10. What effect has a negative sign preceding brackets? (VIII and 26)
11. When quantities are removed into brackets, preceded by the sign —, what must be done with them? (IX and 26)
12. What is the rule for subtraction? (28)

13. Why must we put units of the same denomination in the same vertical column? (24)
14. When a digit in the subtrahend is greater than the corresponding digit in the minuend, what is done? (27 Example 8, or 29)
15. What other plan may be adopted? (30)
16. Upon what principle does this plan proceed? (XI.)
17. Why do we begin to subtract at the right-hand side? (X.)
18. How do we prove subtraction? (31)
19. Upon what principles are these methods of proof founded? (31)
20. Illustrate the difference between *spelling* and *reading* in subtraction. (32)

MULTIPLICATION.

33. Multiplication is a short process of taking one number as many times as there are units in another. Hence multiplication is a short method of performing addition.

34. The number to be taken or multiplied is called the *multiplicand*, and in addition would be called an *addend.*

35. The number denoting how many times the multiplicand is to be taken, or, in other words, that by which we multiply, is called the *multiplier.*

36. The number arising from taking the multiplicand as many times as there are units in the multiplier, is called the *product*, and corresponds to the *sum of the addends* in addition.

The multiplicand and multiplier are called the *factors* of the product because they *make* or *produce* it, (Lat. *factor*, "a maker, agent, or producer.")

37. A prime number is one which cannot be exactly divided by any *whole* number, except the unit *one* and *itself.*

38. A composite number is the product of two or more integral factors, *neither* of which is unity. Thus 16 is a composite number, and its factors are 8 and 2, or 4 and 4.

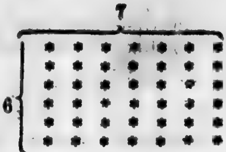
39. Since the product is the result which arises from taking the multiplicand as many times as there are units in the multiplier, it follows:

1st. If the multiplier be equal to unity, the product will be equal to the multiplicand.

2nd. If the multiplier be greater than unity, the product will be as many times greater than the multiplicand as the multiplier is greater than unity.

3d. If the multiplier be less than unity, that is, if it be a proper fraction, the product will be as many times *less* than the multiplicand as the multiplier is less than unity.

40. Let it be required to multiply any two numbers together, say 7 and 6.

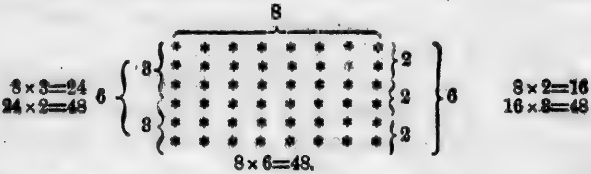


If we make in a horizontal line as many stars as there are units in the multiplicand, and make as many such lines of stars as there are units in the multiplier, it is manifest that the entire number of stars will represent the number of units which result from taking the multiplicand as many times as there are units in the multiplier.

But it is evident that we may consider the 42 stars in the above figure, either as 7 stars taken 6 times, or as 6 stars taken 7 times, that is, 6 × 7=42=7 × 6.

Hence either of the factors may be used as multiplier without altering the product.

41. Let it be required to multiply the number 8 by the composite number 6, of which the factors are 3 and 2.



8 × 6=48.

If we write 8 stars in a horizontal line and make 6 such lines, we shall evidently have in all 8 × 6=48, the number of units in all the lines.

But we may consider the 6 lines as 2 sets of 3 lines each, and in each set of 3 lines there are 8 × 3=24 units. Therefore in the 2 sets there are 24 × 2=48 units. Again we may consider the 6 lines as 3 sets of 2 lines each, and in each set of 2 lines there are 8 × 2=16 units. Therefore in 3 such sets there are 16 × 3=48 units.

Hence 8 × 6=48
8 × 3=24 and 24 × 2=48=8 × 6
8 × 2=16 and 16 × 3=48=8 × 6

And as the same may be shewn for any other composite number as well as for 6, we may conclude that,

When the multiplier is a composite number we may multiply by each of the factors in succession, and the last product will be the entire product sought.

42. As the multiplication of the higher numbers may be resolved into the multiplication of one digit by another, the pupil should make himself perfectly familiar with the following table:

This table is called the Multiplication Table, and was calculated by Pythagoras, a celebrated Greek philosopher who flourished about 500 years before Christ. It was calculated after the following manner:—2 and 2 are 4—twice 2 are 4; 3 and 3 are 6—twice 3 are 6; 4 and 4 are 8—twice 4 are 8, &c.

MULTIPLICATION TABLE.

Twice	3 times	4 times	5 times	6 times	7 times
1 are 2	1 are 3	1 are 4	1 are 5	1 are 6	1 are 7
2 — 4	2 — 6	2 — 8	2 — 10	2 — 12	2 — 14
3 — 6	3 — 9	3 — 12	3 — 15	3 — 18	3 — 21
4 — 8	4 — 12	4 — 16	4 — 20	4 — 24	4 — 28
5 — 10	5 — 15	5 — 20	5 — 25	5 — 30	5 — 35
6 — 12	6 — 18	6 — 24	6 — 30	6 — 36	6 — 42
7 — 14	7 — 21	7 — 28	7 — 35	7 — 42	7 — 49
8 — 16	8 — 24	8 — 32	8 — 40	8 — 48	8 — 56
9 — 18	9 — 27	9 — 36	9 — 45	9 — 54	9 — 63
10 — 20	10 — 30	10 — 40	10 — 50	10 — 60	10 — 70
11 — 22	11 — 33	11 — 44	11 — 55	11 — 66	11 — 77
12 — 24	12 — 36	12 — 48	12 — 60	12 — 72	12 — 84

8 times	9 times	10 times	11 times	12 times
1 are 8	1 are 9	1 are 10	1 are 11	1 are 12
2 — 16	2 — 18	2 — 20	2 — 22	2 — 24
3 — 24	3 — 27	3 — 30	3 — 33	3 — 36
4 — 32	4 — 36	4 — 40	4 — 44	4 — 48
5 — 40	5 — 45	5 — 50	5 — 55	5 — 60
6 — 48	6 — 54	6 — 60	6 — 66	6 — 72
7 — 56	7 — 63	7 — 70	7 — 77	7 — 84
8 — 64	8 — 72	8 — 80	8 — 88	8 — 96
9 — 72	9 — 81	9 — 90	9 — 99	9 — 108
10 — 80	10 — 90	10 — 100	10 — 110	10 — 120
11 — 88	11 — 99	11 — 110	11 — 121	11 — 132
12 — 96	12 — 108	12 — 120	12 — 132	12 — 144

It appears from this table, that the multiplication of the same two numbers in whatever order taken, produce the same product.

NOTE.—Though the part of the multiplication table given above is enough for the pupil to commit to memory at first; yet, after he has made some proficiency in arithmetic, he may find it advantageous to commit what follows, as it will enable him, in many cases, to shorten his work in a considerable degree. The labour of committing a still more extended table would be scarcely compensated by the advantage resulting.

13 times	14 times	15 times	16 times	17 times	18 times	19 times
2 are 26	2 are 28	2 are 30	2 are 32	2 are 34	2 are 36	2 are 38
3 — 39	3 — 42	3 — 45	3 — 48	3 — 51	3 — 54	3 — 57
4 — 52	4 — 56	4 — 60	4 — 64	4 — 68	4 — 72	4 — 76
5 — 65	5 — 70	5 — 75	5 — 80	5 — 85	5 — 90	5 — 95
6 — 78	6 — 84	6 — 90	6 — 96	6 — 102	6 — 108	6 — 114
7 — 91	7 — 98	7 — 105	7 — 112	7 — 119	7 — 126	6 — 133
8 — 104	8 — 112	8 — 120	8 — 128	8 — 136	8 — 144	7 — 152
9 — 117	9 — 126	9 — 135	9 — 144	9 — 153	9 — 162	9 — 171

43. The multiplication of one quantity by another is expressed by ×; thus $7\times9=63$, means that 7 multiplied by 9 is equal to 63.

44. Quantities connected by the sign of multiplication are multiplied by any number, if we multiply any one of the factors by that number; thus $(9\times10\times2)\times27=9\times10\times54$, or $9\times270\times2$; that is, if we multiply the factor 2 or the factor 10 by 27, we, in effect, multiply the whole number $(9\times10\times2)$ by 27.

45. When a quantity within brackets, consisting of several terms connected by the signs + and −, is to be multiplied by any number, each of its parts or terms must be multiplied. This arises from the fact that we consider the several terms within the bracket as constituting but one quantity, and to multiply the whole, we must multiply each of its parts. Thus $(7+8-3)\times3=7\times3+8\times3-3\times3$; and $(8+7-5)\times(13-2)$ means that each of the terms within the former bracket is to be multiplied by each of the terms within the latter, or by their difference.

46. Let it be required to multiply 768 by 9.

Now $768\times9=(700+60+8)\times9=700\times9+60\times9+8\times9$ (Art. 45). Hence so far as the result is concerned, it matters not whether we commence multiplying at the lowest or at the highest denomination; $700\times9+60\times9+8\times9$ being evidently equal to $8\times9+60\times9+700\times9$.

Commencing the multiplication at the left-hand side, or highest denomination, the work is follows :—

OPERATION.

$$\begin{array}{r} 768\\ 9\\ \hline 6300\\ 540\\ 72\\ \hline 6912 \end{array}$$

which may be thus abbreviated.

$$\begin{array}{l} \;\;768\\ \;\;\;\;\;9\\ \hline 63\\ \;\;54\\ \;\;\;\;72\\ \hline 6912 \end{array}$$

Explanation.—7 hundreds multiplied by 9, or taken 9 times. are 63 hundreds: 6 tens multiplied by 9, are 54 tens; and 8 units multiplied by 9, are 72 units. 63 hundreds, 54 tens, and 72 units, added together, make 6912. The second operation shows the only abbreviation possible when we commence at the highest denomination.

Let us now take the same question and commence at the right-hand or lowest denomination.

OPERATION.

I.

$$\begin{array}{r} 768\\ 9\\ \hline 72\\ 540\\ 6300\\ \hline 6912 \end{array}$$

which may be thus abbreviated.

II.

$$\begin{array}{l} \;\;768\\ \;\;\;\;\;9\\ \hline \;\;\;\;72\\ \;\;54\\ 63\\ \hline 6912 \end{array}$$

and thus still farther abbreviated.

III.

$$\begin{array}{r} 768\\ 9\\ \hline 6912 \end{array}$$

Explanation.—No. II. differs from No. I. only in having the unnecessary 0's omitted. In No. III. the principle of *carrying* is taken advantage of, thus —8 units, multiplied by 9, are 72 units, equal to 2 units and 7 tens to carry—6 tens, multiplied by 9, are 54 tens, and 7 tens, make 61 tens, equal to 1 ten, and 6 hundreds to carry; 7 hundreds, multiplied by 9, are 63 hundreds, and 6 hundreds, make 69 hundreds, equal to 6 thousands and 9 hundreds.

Hence, in order that we may be enabled to take advantage of the principle of CARRYING, *we commence the multiplication at the right-hand or lowest denomination.*

47. From the last article (46), for multiplying by any integral multiplier, not exceeding 12, (or 20 if the extended Multiplication Table be used) we deduce the following :—

RULE.

Multiply every order of units in the multiplicand in succession beginning with the lowest, by the multiplier, and divide each product, *so formed, by the number of that denomination which makes one unit of the next higher; write down each remainder under units of its own order, and carry the quotient to the next product.*

EXAMPLE 1.—Multiply $7896·43 by 11.

OPERATION.
$7896·43
11
$86860·73

EXPLANATION.—3 hundredths of dollars, or cents, multiplied by 11, make 33 hundredths, equal to 3 hundredths, to set down, and 3 tenths to carry; 4 tenths of dollars, or tens of cents, multiplied by 11, make 44 tenths of dollars, and 3 tenths we carried, make 47 tenths, equal to 7 tenths, and 4 units to carry; 6 units, multiplied by 11, make 66 units, and 4 units we carried, make 70 units, equal to 0 units to set down, and 7 tens to carry; 9 tens, multiplied by 11, make 99 tens, and 7 tens, make 106 tens, equal to 6 tens and 10 hundreds; 8 hundreds, multiplied by 11, make 88 hundreds, and 10, make 98 hundreds, equal to 8 hundreds and 9 thousands; 7 thousands, multiplied by 11, make 77 thousands, and 9, make 86 thousands, equal to 6 thousands and 8 tens of thousands.

EXAMPLE 2.—Multiply 3 cwt. 2 qrs. 11 lbs. 7 oz. 6 drs. by 7.

OPERATION.

cwt.	qrs.	lbs.	oz.	dr.
3	2	11	7	6
				7
25	1	5	3	10

EXPLANATION.—7 times 6 drams are 42 drams, equal to 10 drams to set down, and 2 oz. to carry; 7 times 7 oz. are 49 oz., and 2 oz., make 51 oz., equal to 3 oz. to set down, and 3 lbs. to carry; 7 times 11 lbs. are 77 lbs., and 3 lbs., make 80 lbs., equal to 5 lbs. to set down, and 3 qrs. to carry; 7 times 2 qrs are 14 qrs., and 3 qrs., make 17 qrs., equal to 1 qr. to set down, and 4 cwt. to carry; 7 times 3 cwt. are 21 cwt., and 4 cwt., make 25 cwt.

EXERCISE 12.

	(1)	(2)	(3)	(4)
Multiply	48960	75460	678000	57800
By	5	9	8	6
	244800			

	(5)	(6)	(7)	(8)
Multiply	5·2736	8·7563	0·21375	0·0067
By	2	4	6	8
	10·5472			

	(9)	(10)	(11)	(12)
Multiply	$767·62	$672·56	$789·76	$578·46
By	2	2	6	5
	$1535·24			

	(13)	(14)	(15)	(16)
Multiply	866342	738579	4716375	8429763
By	11	12	11	12

17. Multiply £32 8s. 6½d. by 5. *Ans.* £162 2s. 8½d.
18. Multiply £43 11s. 4¾d. by 8. *Ans.* £348 11s. 2d.

19. Multiply £125 13s. 0¼d. by 12. *Ans.* £1507 16s. 3d.
20. Multiply 10 cwt. 3 qrs. 5 lbs. by 3. *Ans.* 32 cwt. 1 qr. 15 lbs.
21. Multiply 7 yds. 3 qrs. 1 na. by 7. *Ans.* 54 yds. 2 qrs. 3 na.
22. Multiply 11 oz. 10 dwt. 19 grs. by 12. *Ans.* 11 lbs. 6 oz. 9 dwt. 12 gr.

48. When the multiplier is a composite number, and can be resolved into two or more factors, neither of which is greater than 12, we deduce from (41) the following:—

RULE.

Multiply by each of the factors in succession and the last product will be the entire product sought.

Example 1.—Multiply 3 hrs. 7 min. 14 sec. by 64.

OPERATION.

hrs.	min.	sec.	×64=8×8
3	7	14	
		8	
1 0	57	52	
		8	
8 7	42	56 *Ans.*	

Explanation.—Multiplying 3 hrs. 7 min. 14 sec. by 8, we obtain 1 day 0 hrs. 57 min 52 sec., which we again multiply by 8, and obtain 8 days 7 hrs. 42 min. 56 sec., which is the product of 3 hrs. 7 min. 14 sec., by 8 times 8 or 64.

Example 2.—Multiply 796·437 by 132.

OPERATION.

796·437×132=11×12
11

8760·807=11 times multiplicand.
12

105129·684=12 times 11 times multiplicand.

Explanation.—We first multiply the given number by eleven, or, in other words, take it 11 times, and then take this result 12 times, which is evidently equivalent to taking the given number 12 times 11 or 132 times.

Example 3.—Multiply 16 cwt. 3 qrs. 11 lbs. by 270.

OPERATION.

cwt.	qrs.	lb.	
16	3	11	×270
		3	
50	2	8	
		9	
455	0	22	
		10	
4552	0	20	

Explanation.—270=10 times 27 or 10×3×9. If, therefore, we take the given multiplicand 3 times, and then this product 9 times and then this second product 10 times, it is evident we shall have, in effect, taken the given multiplicand 3×9×10 or 270 times.

Exercise 13.

1. Multiply $169·78 by 36. *Ans.* $6112·08.
2. Multiply $796342·3 by 121. *Ans.* 96357418·3.
3. Multiply $33460 by 144. *Ans.* $4818240.
4. Multiply 735 by 648. *Ans.* 476280.
5. Multiply £3 7s. 6d. by 18. *Ans.* £60 15s. 0d.

6. Multiply £5 14s. 6½d. by 22. *Ans.* £125 19s. 11d.
7. Multiply £3 4s. 7d. by 310. *Ans.* £2615 12s. 6d.
8. Multiply 11 cwt. 3 qrs. 14 lb. 7 o[illegible].
Ans. [illegible]42 cwt. 1 qr. 4 lbs. 10 oz.
9. Multiply 26 bush. 3 pks. 1 gal. 1 qt. 1 pt. by 49.
Ans. 1319 bush. 0 pks. 1 gal. 1 qt. 1 pt.
10. Multiply 2 yds. 2 qrs. 2 na. 2 in. by 68.
Ans. 168 yds. 3 qrs. 2 na. 0 in.
11. Multiply 5 days 17 hrs. 33 min. 11 sec. by 288.
Ans. 1650 days, 15 hrs. 16 min. 48 sec.

49. When the multiplicand is a denominate number and the multiplier is greater than 12, but not a composite number, we proceed according to the following:—

RULE.

Take the nearest composite number to the given multiplier, multiply successively by its factors, and add to or subtract from the product so many times the multiplicand as the assumed composite number is less or greater than the given multiplier.

EXAMPLE 1.—Multiply £62 12s. 6d. by 76.

```
OPERATION.
  £     s.  d.
  62   12   6
            8
 ------------
 501    0   0
            9
 ------------
4509    0   0 = 72 times multiplicand.
 250   10   0 =  4 times multiplicand.
 ------------
£4759  10   0 = 76 times multiplicand.
```

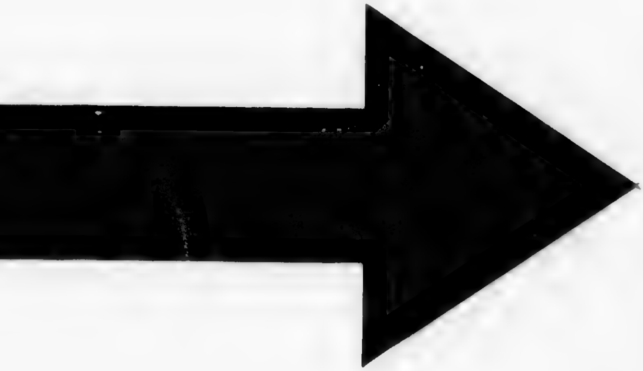
EXPLANATION.—We take 76=9×8+4, and thus we get 72 times the multiplicand, and to it adding 4 times the multiplicand, obtain the desired product, viz., 76 times the multiplicand.

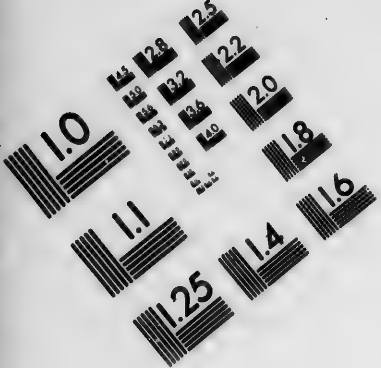
Instead of multiplying as above, we might have multiplied by 7 and 10 and increased the result by 6 times the multiplicand, or we might have multiplied by 7 and 11, and decreased the result by once the multiplicand, &c.

EXAMPLE 2.—Multiply 17 lbs. 8 oz. 7 dr. 2 scr. 16 grs. by 789.

```
                    OPERATION.
  lbs.   oz.   dr.   scr.   grs.
   17     8     7     2     16 × 9 =   9 times multiplicand.
                            10
  -----------------------------
  178     8     7     1      0 × 8 =  80 times multiplicand.
                            10
  -----------------------------
 1783     8     1     1      0
                             7
  -----------------------------
12182    10     1     1      0 = 700 times multiplicand.
 1386     7     2     2      0 =  80 times multiplicand.
  155    11     7     1      4 =   9 times multiplicand.
  -----------------------------
13675     5     2     1      4 = 789 times multiplicand.
```

[redacted]





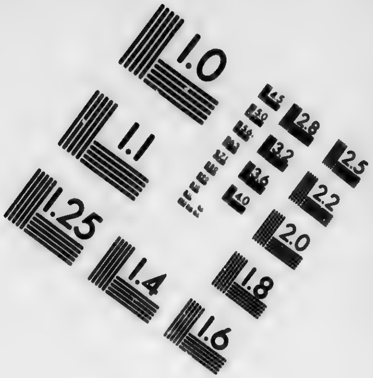
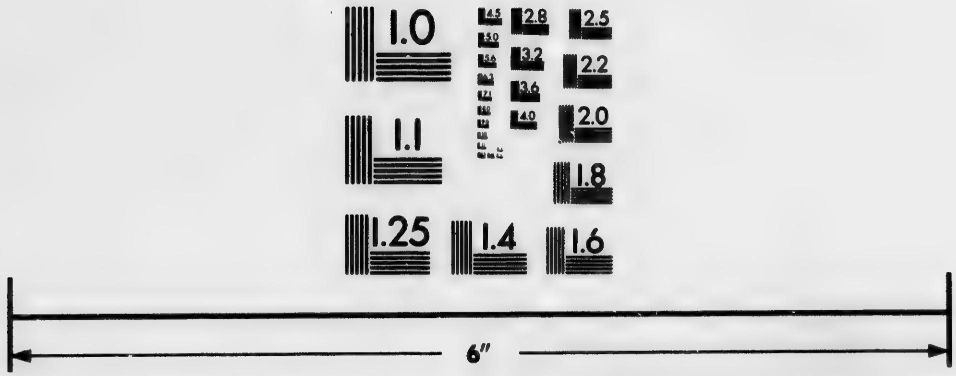
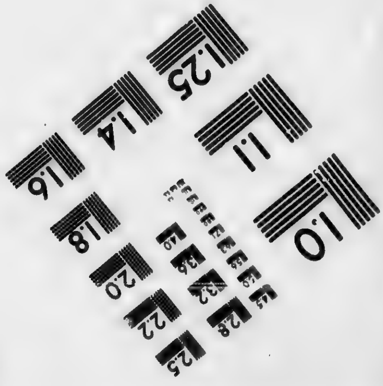


IMAGE EVALUATION
TEST TARGET (MT-3)





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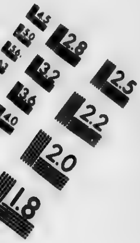
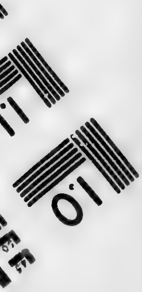


4.5
5.0
5.6
2.8
3.2
2.5
3.6
2.2
4.0
2.0
1.8



1.1
1.0

Explanation.—We divide the given multiplier into 700+30+9, and obtain the 3 partial products, which we add together, for the entire product.

Example 3.—Multiply 3 wks. 6 days 17 hrs. 21 min. 12 sec. by 4736.

OPERATION.

wks.	da.	h.	min.	sec.		wks.	da.	h.	min.	sec.	
3	6	17	21	12	× 6 =	23	5	8	[illegible]	[illegible]	= 6 times multiplicand.
				10							
39	4	5	32	0	× 3 =	118	5	16	36	0	= 30 times multiplicand.
				10							
396	0	7	20	0	× 7 =	2772	2	3	20	0	= 700 times multiplicand.
				10							
3960	3	1	20	0	× 4 =	15841	5	5	20	0	= 4000 times multiplicand.
					Ans.	18756	4	9	23	12	= 4736 times multiplicand.

Example 4.—Multiply £47 16s. 2d. by 5783.

5783 = 5×1000+7×100+8×10+3.

OPERATION.

£	s.	d.		£	s.	d.	
47	16	2	× 3 =	143	8	6	= product by units of the multiplicand.
		10					
478	1	8	× 8 =	3824	13	4	= product by tens of the multiplicand.
		10					
4780	16	8	× 7 =	33465	16	8	= product by hundreds of the multiplicand.
		10					
47808	6	8	× 5 =	239041	13	4	= product by thousands of the multiplicand.
			Ans.	276475	11	10	= product by entire multiplier.

Exercise 14.

1. Multiply £12 2s. 4d. by 83. *Ans.* £1005 13s. 8d.
2. Multiply £963 0s. 0¾d. by 999. *Ans.* 962040 2s. 5¼d.
3. Multiply £3 6s. 5¼d. by 3178. *Ans.* £10556 18s. 4½d.
4. Multiply 16 bush. 3 pks. 1 gal. by 678. *Ans.* 11441 bush. 1 pk. 0 gal.
5. Multiply 23 m. 6 fur. 33 rds. 4 yds. by 247. *Ans.* 5892 m. 2 fur. 10 rds. 3½ yds.
6. Multiply 3 S. 16° 30′ 45″ by 721. *Ans.* 2559 S. 25° 30′ 45″.

50. It may be proper here to caution the pupil against the absurd attempt to multiply one denominate number by another. Multiplication is merely a particular kind of addition, and when we are required to multiply a quantity by any number, we are simply required to repeat it as many times as there are units in the multiplier. It is evident, then, that to talk of multiplying £19 19s. 11¾d., by £19 19s. 11¾d., or, in other words, of adding or repeating £19 19s. 11¾d. £19 19s. 11¾d. times, is simply ridiculous. Nevertheless, great pains have been taken to show that 2s. 6d. may be multiplied by 2s. 6d., and that the product will be either 3¾d. or 6s. 3d.!! Undoubtedly, 2s. 6d. can be taken 2½ times, and the result will be 6s. 3d.; or it can be taken one-eighth of a time, and the result will be 3¾d.; but this is a very different thing from taking it 2s. 6d. times. In fact it is quite as nonsensical to talk of taking

2s. 6d. 2s. 6d. times as it would be to talk of taking 6 lbs. of beef 6 lbs. of beef times; or, 7 bars of music 7 bars of music times, &c. Duodecimal multiplication, which is sometimes adduced, as a proof that one denominate number can be multiplied by another, affords no support whatever to the theory, as will be fully shown hereafter. (See Sec. III.)

51. Let it be required to multiply 729 by 478.

OPERATION.
729
478
5832
5103
2916
348462

Explanation.—From the preceding examples it is evident that when units are multiplied into any order whatever, the product will always be of that order. Here, then, we first multiply by the 8 units, as in (47) Next we multiply by the 7 tens, thus:—9 units, multiplied by 7 tens, give 63 tens, equal to 3 tens, which we set down in the column of tens, and 6 hundreds, which we carry; 2 tens, multiplied by 7 tens, give 14 hundreds, and 6 hundreds which we carried, make twenty hundreds, equal to 0 hundreds to set down and 2 thousands to carry, &c. Next we multiply by the 4 hundreds as follows:—9 units multiplied by 4 hundreds, give 36 hundreds, equal to six hundreds to set down in the hundreds column, and 3 thousands to carry, &c. Lastly, we add the several partial products together.

Hence, when the multiplicand is an abstract number, the multiplier being greater than 12 and not a composite number, we have the following:—

RULE.

Multiply the multiplicand by each figure of the multiplier separately, beginning with the lowest, and write the partial products in separate lines, placing the first figure of each line directly under the figure by which you multiply, and, lastly, add the several partial products together.

Example.—Multiply 7423 by 6709.

OPERATION.
7423
6709
66807
519610
44538
49800807

Explanation.—Here, as there are no tens in the multiplier, we may either proceed directly to the hundreds after multiplying by the units, or we may set down a 0 under the tens, and then write the product by the hundreds in the same line, always remembering to place the first digit of the partial product under the figure by which we are multiplying in order that all the digits of the same order may come in the same vertical column.

Exercise 15.

	(1)	(2)	(3)	(4)	(5)
Multiply	325	765	732	997	667
By	95	765	456	345	347

6. Multiply 7071 by 556. *Ans.* 3931476.
7. Multiply 15607 by 3094. *Ans.* 48288058.
8. Multiply 39948123 by 6007. *Ans.* 239968374861.
9. Multiply 2778588 by 9867. *Ans.* 27416327796.

52. Let it be required to multiply 63·5 by ·97.

OPERATION.
63·5
·97
4·445
57·15
61·595

EXPLANATION.—Since (51) any order, multiplied by units, will give that order—tenths, multiplied by units, will give tenths. Hence it is obvious that tenths, multiplied by tenths will give the next lower order, or hundredths; and also that tenths, multiplied by hundredths, will give the next lower order again, or thousandths. In the above example, therefore, we proceed thus.—5 tenths, multiplied by 7 hundredths, give 35 thousandths, equal to 5 thousandths to set down and 3 hundredths to carry; 3 units, multiplied by 7 hundredths, give 21 hundredths, and three hundredths we carried, make 24 hundredths, equal to 4 hundredths to set down and 2 tenths to carry; 6 tens, multiplied by 7 hundredths, give 42 tenths; and 2 tenths we carried, make 44 tenths, equal to 4 tenths and 4 units. Again, 5 tenths, multiplied by 9 tenths, give 45 hundredths, equal to 5 hundredths to set down and 4 tenths to carry, &c.

53. Strictly speaking, all examples in multiplication of decimals should be worked according to the above method. An attentive consideration of the reasonings in (52) will, however, show that the lowest digit of the product of any two numbers containing decimals, must always be a number of places to the right of the decimal point, equal to the *sum* of the decimal places, in both multiplicand and multiplier.

Hence, when the multiplicand or multiplier, or both, contain decimals, we deduce the following—

RULE.

Multiply as though there were no decimals, and then remove the decimal point in the product as many places to the left as there are decimals in both the multiplicand and the multiplier.

EXAMPLE 1.—Multiply 5·63 by 0·00005.

OPERATION.
563
5
2815
Ans. ·0002815

EXPLANATION.—We multiply 563 by 5, and remove the decimal point *seven* places to the left, since there are *five* decimal places in the multiplier and *two* in the multiplicand, that is, we have taken a number a hundred times too great a hundred thousand times too often, and the product 2815 is therefore ten million times too great, and to make it what it should be, we divide it by ten millions; or, in other words, remove the decimal point seven places to the left.

EXAMPLE 2.—Multiply 2·078 by 5·12.

OPERATION.
2·078
5·12
4156
2078
10390
10·63936

EXPLANATION.—We multiply as though both were whole numbers and cut off *five* decimals, since there are *three* in the multiplicand and *two* in the multiplier.

EXERCISE 16.

	(1)	(2)	(3)
Multiply	·003296	41·78	86·1234
By	5·782	·0629	2·0006
Product	·019057472	2·627962	

4. Multiply 3·2517 by ·023. *Ans.* ·0747891.
5. Multiply 64·001 by 340. *Ans.* 21760·34.
6. Multiply 482000 by ·37. *Ans.* 178340.
7. Multiply 3782·4 by ·00917. *Ans.* 34·684608.
8. Multiply 87·96 by 220. *Ans.* 19351·2.

PROOF OF MULTIPLICATION.

54. *If the multiplier is not greater than* 12, *multiply the multiplicand by the multiplier, minus one, and add the multiplicand to the product. The sum should be the same as the product of the multiplicand by the whole multiplier.*

If the multiplier be greater than 12 and the multiplicand an abstract number:—

FIRST METHOD.—*Multiply the multiplier by the multiplicand, and if the product thus obtained agree with the other, the work may be considered correct.*

This method of proof depends upon the principle (40) that the product of two numbers is the same whichever is taken as multiplier.

SECOND METHOD.—*Divide the product by one of the factors, and if the quotient thus obtained is equal to the other factor, the work is correct.*

This is simply reversing the operation, i. e., breaking up the product into its factors.

THIRD METHOD.—*Divide the sum of the digits of the multiplicand by* 9 *and set down the remainder; divide also the sum of the digits of the multiplier by* 9 *and set down the remainder; multiply these two remainders together, divide the sum of the digits in their product by* 9, *and if the remainder thus obtained is equal to the remainder obtained by dividing the sum of the digits in the product of the multiplicand and the multiplier by* 9, *the work is generally correct: if these two last remainders are different, it must be wrong.*

EXAMPLE 1.—Let the quantities multiplied be 9426 and 3785.

Taking the nines from 9426, we get 3 as remainder.
And from 3785, we get 5.

```
   47130
   75408   3 × 5 = 15, from which 9 being taken, 6 are left.
  65982
 28278
```

Taking the nines from 35677410, 6 are left.

The remainders being equal, we are to presume the multiplication is correct. The same result, however, would have been obtained even if we had displaced digits, added or omitted cyphers, or fallen into errors which had counteracted each other; but, with ordinary care, none of these are likely to occur.

Example 2.—Let the numbers be 76542 and 8436.

Taking the nines from 76542, the remainder is 6.
Taking them from 8436, it is 3.

$$\begin{array}{r} 459252 \\ 229626 \\ 306168 \\ 612336 \\ \hline \end{array} \quad 6 \times 3 = 18, \text{ the remainder from which is } 0.$$

Taking the nines from 645708312 also, the remainder is 0.

The remainders being the same, the multiplication may be considered correct.

Note.—This proof applies, whatever may be the position of the decimal point in either of the given numbers.

Example 3.—Let the numbers be 4·63 and 5·4.

From 4·63, the remainder is 4.
From 5·4, it is 0.

$$\begin{array}{r} 1852 \\ 2315 \\ \hline \end{array} \quad 4 \times 0 = 0, \text{ from which the remainder is } 0.$$

From 25·002 the remainder is 0.

55. The principle on which this process depends is, that if any number is divided by 9, and the sum of its digits also be divided by 9, the remainders are, in both cases, the same.

Thus taking the number 7825, we have:

$$\begin{aligned} \frac{7825}{9} &= \frac{7000+800+20+5}{9} = \frac{7000}{9}+\frac{800}{9}+\frac{20}{9}+\frac{5}{9} \\ &= 7 \times \frac{1000}{9} + 8 \times \frac{100}{9} + 2 \times \frac{10}{9} + \frac{5}{9} \\ &= 7 \times (111 + \tfrac{1}{9}) + 8 \times (11 + \tfrac{1}{9}) + 2 \times (1 + \tfrac{1}{9}) + \tfrac{5}{9} \\ &= 777 + \tfrac{7}{9} + 88 + \tfrac{8}{9} + 2 + \tfrac{2}{9} + \tfrac{5}{9} \\ &= 777 + 88 + 2 + \tfrac{7}{9} + \tfrac{8}{9} + \tfrac{2}{9} + \tfrac{5}{9} \\ &= 777 + 88 + 2 + \frac{7+8+2+5}{9} \end{aligned}$$

Hence the remainder arising from the division of 7825 by 9 is evidently the same as that arising from dividing 7+8+2+5 or 22, which is the sum of its digits, by 9.

56. Casting the nines from the factors, multiplying the resulting remainders, and casting the nines from the product, will leave the same remainder as if the nines were cast from the product of the factors—provided the multiplication has been correctly performed.

Thus, let the factors be 578 and 464.

Casting the nines from 5+7+8 (which we have just seen is the same as casting the nines from 578), we obtain 2 as *remainder*. Casting the nines from 4+6+4, we get 5 as *remainder*. Multiplying 2 and 5 we obtain 10 as product, which, when the nines are taken away, will give 1 as a remainder.

We can show that 8 will be the remainder, also, if we cast the nines from the product of the factors;—which is effected by setting down this product, and taking, in succession, quantities that are equal to it—as follows:—

$$578 \times 464 = \text{(the product of the factors)}.$$
$$= (5 \times 100 + 7 \times 10 + 8) \times (4 \times 100 + 6 \times 10 + 4)$$
$$= \left\{ 5 \times (99+1) + 7 \times (9+1) + 8 \right\} \times \left\{ 4 \times (99+1) + 6 \times (9 \times 1) + 4 \right\}$$
$$= (5 \times 99 + 5 + 7 \times 9 + 7 + 8) \times (4 \times 99 + 4 + 6 \times 9 + 6 + 4.)$$

5×99 expresses a number of nines: it will continue to do so when multiplied by all the quantities within the second brackets, and is, therefore, to be cast out; and, for a similar reason, 7×9. Again 4×99 expresses a number of nines; it will continue to do so when multiplied by the quantities within the first brackets, and is, therefore, to be cast out; and for a similar reason, 6×9. There will then be left only (5+7+8)×(4+6+4)—from which the nines are still to be cast out, the *remainders* to be multiplied together, and the nines to be cast from their product;—but we have done all this already, and obtained 8 as remainder.

CONTRACTIONS IN MULTIPLICATION.

57. I. To multiply by 5:

Affix a 0 to the multiplicand and divide the result by 2.

Reason $5 = \frac{10}{2}$.

II. To multiply by 15:

Affix a 0 to the multiplicand and to the result add half of itself.

Reason $15 = 10 + \frac{10}{2}$.

III. To multiply by 25:

Affix two 0s to the multiplicand and divide the result by 4.

Reason $25 = \frac{100}{4}$.

IV. To multiply by 125:

Affix three 0s to the multiplicand and divide the result by 8.

Reason $125 = \frac{1000}{8}$.

V. To multiply by 75:

Affix two 0s to the multiplicand and from the result take one-fourth of itself.

Reason $75 = 100 - \frac{100}{4}$.

VI. To multiply by 175:

Affix two 0s—multiply the result by 7 and divide by 4.

Reason $175 = \frac{700}{4}$.

VII. To multiply by 275:

Affix two 0s—multiply the result by 11 and divide by 4.

Reason $275 = \frac{1100}{4}$.

VIII. To multiply by 13, 14, 15, &c.; or by 1 with either of the other digits *affixed* to it:

Example.
2325 × 13
6975
Ans. 30225

Multiply by the units' figure of the multiplier, and write each figure of the partial product one place to the right of that from which it arises; finally, add the partial product to the multiplicand, and the result will be the answer required.

Reason.—This is the same in effect as if we actually multiplied by the common method. We merely make the multiplicand serve for the second partial product.

IX. To multiply by 21, 31, 41, &c., or by 1 with either of the other significant figures *prefixed* to it:

Example.
365 × 21
730
Ans. 7665

Multiply by the tens' figure of the multiplier, and write the first figure of the partial product in the tens' place; finally, add this partial product to the multiplicand, and the result will be the answer required.

Reason.—The reason of this method of contraction is substantially the same as that of the preceding.

X. To multiply by 101, 102, 103, 104, &c., or by 10 with either of the other digits *affixed* to it:

Multiply by the units' figure of the multiplier and write the partial product, thus obtained, two places to the right of the multiplicand; finally, add the partial product to the multiplicand.

Reason.—Substantially the same as No. 8.

XI. To multiply by any number of nines:

Remove the decimal point of the multiplicand, so many places to the right (by affixing 0's if necessary) as there are nines in the multiplier; and subtract the multiplicand from the result.

Example 1.—Multiply 7347 by 999.

7347 × 999 = 7347000 − 7347 = 7339653.

We, in such a case, merely multiply by the next higher convenient composite number, and subtract the multiplicand as many times as we have taken it too often; thus, in the example just given—

7347 × 999 = 7347 × (1000 − 1) = 7347000 − 7347 = 7339653.

Example 2.—Multiply 678943 by 999999.

678943 × 1000000 = 678943000000
678943 × 1 = 678943

678943 × 999999 = 678942321057

Example 3.—Multiply 78·9645 by 99993.

78·9645 × 100000 = 7896450
78·9645 × 7 = 552·7515

78·9645 × 99993 = 7895897·2485

XII. When it is not necessary to have as many decimal places in the product, as are in both multiplicand and multiplier—

Reverse the multiplier, putting its units' place under the place of that denomination in the multiplicand, which is the lowest of the required product.

Multiply by each digit of the multiplier beginning with the denomination over it in the multiplicand; but adding what would have been obtained, on multiplying the preceding digit of the multiplicand—unity, if the number obtained would be between 5 and 15; 2, if between 15 and 25; 3, if between 25 and 35, &c.

Let the lowest denominations of the products, arising from the different digits of the multiplicand, stand in the same vertical column.

Add up all the products for the total product; from which cut off the required number of decimal places.

EXAMPLE 1.—Multiply 5·6784 by 9·7324, so as to have four decimals in the product.

Short method.	Ordinary method.
56784	5·6784
42379	9·7324
511056	22 | 7136
39749	113 | 568
1703	1703 | 52
113	39748 | 8
22	511056
55·2643	55·2644 | 6016

9 in the multiplier expresses units; it is therefore put under the *fourth* decimal place of the multiplicand—that being the place of the lowest decimal required in the product.

In multiplying by each succeeding digit of the multiplier we neglect an additional digit of the multiplicand; because, as the multiplier decreases, the number multiplied must increase—to keep the lowest denomination of the different products, the same as the lowest denomination required in the *total* product. In the example given, 7 (the second digit of the multiplier) multiplied by 8 (the second digit of the multiplicand) will evidently produce the same denomination as 9 (one denomination higher than the 7), multiplied by 4 (one denomination lower than the 8). Were we to multiply the lowest denomination of the multiplicand by 7, we should get (28) a result in the *fifth* place to the right of the decimal point; which is a denomination supposed to be, in the present instance, too inconsiderable for notice—since we are to have only *four* decimals in the product. But we add unity for every ten that would arise, from the multiplication of an additional digit of the multiplicand; since every *ten* constitutes *one* in the lowest denomination of the required product. When the multiplication of an additional digit of the multiplicand would give *more* than 5, and *less* than 15, it is nearer to the truth to suppose we have ten than either 0 or 20; and therefore it is more correct to add 1 than either 0 or 2. When it would give more than 15 and less than 25, it is nearer to the truth to suppose we have 20, than either 10 or 30; and therefore it is more correct to add 2 than 1 or 3; &c. We may consider 5 *either* as 0 or 10; 15 *either* as 10 or 20; &c.

On inspecting the results obtained by the abridged, and ordinary methods, the difference is perceived to be inconsiderable. When greater accuracy is desired, we should proceed as if we intended to have more decimals in the product, and afterwards reject those that are unnecessary.

Example 2.—Multiply 8·76532 by 0·5764, so as to have three decimal places.

```
 876532
4675
-------
4383
 613
  52
   3
----
5·051
```

There are no units in the multiplier; but, as the rule directs, we put its units' *place* under the third decimal place of the multiplicand. In multiplying by 4, since there is no digit over it in the multiplicand, we merely set down what would have resulted from the multiplying the preceding denomination of the multiplicand.

Example 3.—Multiply 0·23257 by 0·243, so as to have four decimal places.

```
 23257
 342
------
 465
  93
   7
-----
0·0565
```

We are obliged to place a cipher in the product to make up the required number of decimals.

Exercise 17.

1. The canals in Canada amount to 216 miles in length, and their average cost was $83469 per mile. What was the total cost of the canals of Canada?

2. The Great Western Railroad is 229 miles in length, and its cost was about $61135·37 per mile. What was the total cost of this road?

3. The Austrian empire contains 255226 square miles, and the population averages 143 per square mile. What is the entire population of the Austrian empire?

4. France contains 203736 square miles, and the population averages 176 per square mile. What is the entire population of France?

5. Great Britain contains 116700 square miles, and the population averages 235 per square mile. What is the entire population of Great Britain?

6. The total number of Common Schools in operation in Canada West, during the year 1857, was 3721; allowing an average of 73 pupils to each, how many children were in attendance at the Common Schools?

7. 32000 seeds have been counted in a single poppy; how many would be found in 297 of these?

8. 9344000 eggs have been found in a single cod fish; how many would there be in 35 such?

9. Multiply 128 lbs. 4 oz. 7 drs. 2 scr. 17 gr. by 749.

10. Multiply 1698732 by 999998.

11. Multiply 128 bush. 1 pk. 1 gal. 1 qt. 1 pt. by 640.

12. What will be the cost of a chest of tea containing 89 lbs. at 73 cents per lb.?

13. How much cloth will it take to make the clothes for a regiment of soldiers containing 1143 men, if each suit requires 7 yds. 3 qrs. 2 na. 1 in.?

14. Multiply 1634·5789 by 635000.

15. A person dying bequeathed the whole of his property to his three sons. To the youngest he gave $968·49; to the second, 8·4 times as much as the youngest; and to the eldest 3·7 times as much as to the second. Required the value of his property.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the articles of the section.*

1. What is multiplication? (33)
2. What is the multiplicand? (34)
3. What is the multiplier? (35)
4. What is the product? (36)
5. Why are the multiplier and multiplicand called the factors of the product? (36)
6. What is a prime number? (37)
7. What is a composite number? (38)
8. If the multiplier be greater than unity, how will the product compare with the multiplicand? (39)
9. If the multiplier be equal to unity, how will the product compare with the multiplicand? (39)
10. If the multiplier be less than unity, how will the product compare with the multiplicand? (39)
11. Show that either of the factors may be used as multiplier without altering the value of the product. (40)
12. Show that when the multiplier is a composite number we may obtain the entire product by multiplying by each of the factors in succession. (41)
13. By whom was the multiplication table calculated? (42)
14. How was it calculated? (42)
15. What is the sign of multiplication? (43)
16. How do we multiply a quantity consisting of several factors connected by the sign of multiplication? (44)
17. How do we multiply a quantity consisting of several terms, connected by the signs + and − enclosed within a bracket? (45)
18. What is meant by (7+3−2+5) × (9+3−7)? (45)
19. Why do we begin multiplying a number at the right-hand side? (46)
20. What is the rule for multiplication when the multiplier is not greater than 12? (47)
21. What is the rule when the multiplier is a composite number, none of its factors being greater than 12? (48)
22. What is the rule when the multiplicand is a denominate number, and the multiplier greater than 12, but not a composite number? (49)
23. Show the absurdity of attempting to multiply one denominate number by another. (50)
24. When the multiplicand is an abstract number, and the multiplier greater than 12, but not a composite number, what is the rule? (51)
25. When the multiplicand or multiplier, or both, contain decimals, what is the rule? (53)
26. Give the reason of this rule. (52 and 53)
27. How do we prove multiplication when the multiplier is less than 12? (52)

28. How do we prove multiplication when the multiplicand is an abstract number and the multiplier is greater than 12? (54)
29. Upon what does the proof by casting out the nines depend? (55)
30. Prove this principle. (55)
31. Prove that casting the nines from the factors, multiplying the resulting remainders, and casting the nines from the product, will leave the same remainder as if the nines were cast from the product of the factors. (56)
32. What short methods have we for multiplying by 5, 25 and 125? (57)
33. What short methods of multiplying by 15 and 75? (57)
34. How may we multiply by 175? How by 275? (57)
35. How may we multiply by 13, 14, 15, &c.? How by 101, 102, 103, &c.? (57)
36. How may we multiply by 21, 31, 41, &c.? (57)
37. How may we multiply by any number of nines? (57)
38. How may we contract the work when we require only a limited number of decimals? (57)

DIVISION.

58. Division is the process of finding how many times one number is contained in another.

59. The number by which we divide is called the *divisor*.

60. The number to be divided is called the *dividend*.

61. The number obtained by division, that is, the number which shows *how many times* the divisor is contained in the dividend is called the *quotient* (Lat. *quoties*, "how many times.")

62. If the divisor be less than the dividend, the quotient will be greater than unity.

If the divisor be equal to the dividend, the quotient will be equal to unity.

If the divisor be greater than the dividend, the quotient will be less than unity.

63. It is sometimes found that the dividend does not contain the divisor an *exact* number of times; in such cases the quantity left after the division is called the *remainder*.

The remainder, being a part of the dividend, is, of course, of the same denomination.

The remainder must be *less* than the divisor—otherwise the divisor would be contained *once more* in the dividend.

64. Division is merely a short method of performing a particular kind of subtraction (Art. 6, Sec. II.) The dividend corresponds to the minuend, the divisor to the subtrahend, and the remainder to the difference. The *quotient* has no corresponding quantity in subtraction—since it simply tells how many times the divisor can be subtracted from the dividend.

It will help us to understand how greatly division abbreviates subtraction, if we consider how long a process would be required to discover—by actually subtracting it—how often 7 is contained in 8563495724, while as we shall find, the same thing can be effected by *division* in less than a minute.

65. Since the quotient shows how many times the dividend contains the divisor, it follows that the divisor and quotient are the *factors* of the dividend. Hence if the divisor and quotient be multiplied together, and the remainder, if any, added to the product, the result will be equal to the dividend.

66. We have three ways of expressing the division of one quantity by another:—

1st. By the sign ÷ written between them; thus, $15 \div 3 = 5$.

2nd. By the sign : written between them; thus, $15 : 3 = 5$.

3rd. By writing the dividend above and the divisor below a horizontal line; thus, $\frac{15}{3} = 5$.

Two quantities written thus $\frac{6}{11}$ constitute what is called a fraction, and the expression is read *six-elevenths.*

It is usual and proper to write the remainder obtained in division, in the form of a fraction; thus $17 \div 3$ gives 5 as a quotient and 2 as a remainder. Now the remainder, 2, is written above the line, and divisor 3 below the line; the whole quotient being expressed thus $5\frac{2}{3}$ (read five and two-thirds); the meaning of which is, that 3 is contained in 17, 5 times and $\frac{2}{3}$ of a time.

67. When a quantity consisting of several terms connected by the sign of multiplication is to be divided, dividing any one of the factors will be the same as dividing the product; thus $5 \times 10 \times 25 \div 5 = \frac{5}{5} \times 10 \times 25$, for each is equal to 250.

68. When a quantity consisting of several terms connected by the signs + and —, contained within brackets, is to be divided, it is necessary, on removing the brackets, to put the divisor under each of the terms of the quantity; thus $(6+3-7+9) \div 3$, or $\frac{6+3-7+9}{3} = \frac{6}{3} + \frac{3}{3} - \frac{7}{3} + \frac{9}{3}$; for we do not divide the whole unless we divide *all* its parts.

69. It will be seen from (68) that the horizontal line which separates the dividend from the divisor assumes the place of a pair of brackets when the dividend consists of several terms; and, therefore, when the quantity to be divided is subtractive, it will sometimes be necessary to change the signs, as already directed (26); thus:

$$\frac{6}{2}+\frac{13-3}{2}=\frac{6+13-3}{2}\text{; but }\frac{27}{3}-\frac{15-6+9}{3}=\frac{27-15+6-9}{3}$$

EXAMPLE 1. Let it be required to divide 798 by 3.

OPERATION.
3)798
266

EXPLANATION.—Place the divisor a little to the left of the dividend and separate them by a short curve line. Also draw a straight line beneath the dividend.

Now $\frac{798}{3}=\frac{700+90+8}{3}=\frac{600+190+8}{3}=\frac{600+180+18}{3}=\frac{600}{3}+\frac{180}{3}+\frac{18}{3}=200+60+6=266$ (See 68).

Instead of going through this long operation it is evident that we may proceed as follows: 3 units into 7 hundreds will go 2 (hundreds) times and leave a remainder 1, which being of the order of hundreds, is equal to 10 tens; 10 tens and 9 tens make 19 tens, and 3 into 19 goes 6 (tens) times and leaves a remainder 1, which, being of the order of tens is equal to 10 units; 10 units and 8 units make 18 units, and 3 units into 18 units goes 6 (units) times.

EXAMPLE 2. Let it be required to divide 917 lb. 13 oz. 12 dr. by 4.

OPERATION.
lb. oz. dr.
4)917 13 12
229 7 7

EXPLANATION.—Placing the dividend and divisor as before, we proceed thus: 4 in 9, 2 (hundreds) times and 1 over; 1 hundred, equal to 10 tens, and 1 ten make 11 tens; 4 in 11, 2 (tens) times and 3 over; 3 tens, equal to 30 units, and 7 units make 37 units; 4 in 37, 9 times and 1 over, which is 1 lb. because the 917 are pounds (68): 1lb., equal to 16oz. and 13oz. make 29 oz., 4 in 29, 7 times and 1 over, which is 1 oz., since the 29 are oz.; 1 oz. is equal to 16 drams and 12 drams make 28 drams; 4 in 28, 7 times.

Observe that any order divided by units gives that order in the quotient.

EXAMPLE 3. Let it be required to divide 9789 by 26.

OPERATION.
26)9789(376
78
198
182
169
156
13 rem.
Ans. $376\frac{13}{26}$

EXPLANATION.—Placing the dividend and divisor as before, we say 26 in 9 (thousands) no times; 26 in 97 (hundreds), 3 (hundreds) times. We place the 3 (hundreds) to the right of the dividend and multiplying the divisor 26 by it, get 78 hundred, which we subtract from the 97 hundred, and obtain a remainder 19 hundreds. 19 hundreds are equal to 190 tens, and 8 tens make 198 tens; 26 in 198, 7 (tens) times. Multiplying the 26 by the 7 tens, we get 182 tens, which, subtracted from 198 tens, leaves a remainder of 16 tens. 16 tens are equal to 160 units and 9 units make 169 units; 26 in 169, goes 6 times, and leaves a remainder 13. This 13 should be divided by 26, but since 13 does not contain 26, the division cannot be effected, and we can only indicate it, which we do by placing the 26 under the 13, as is explained in (Art. 60).

The complete quotient is therefore $376\frac{13}{26}$ read 376 and thirteen-twenty-sixths or 376 and 13 divided by 26.

71. From the preceding illustration and example we deduce, for the division of numbers, the following general

RULE.

Beginning with the highest order of units in the dividend, pass on to the lower orders until the fewest number of figures be found that will contain the divisor; divide these figures by it, for the first figure of the quotient; this figure will be of the same order as that of the lowest used in the partial dividend.

Multiply the divisor by the quotient figure so found, and subtract the product from the dividend, being careful to place units of the same order in the same vertical column. Reduce the remainder to units of the next lower order, and add in the units of that order found in the dividend: this will furnish a new dividend.

Proceed in a similar manner until units of every order shall have been divided.

Example 1.—Divide 98765 by 7.

OPERATION.
7)98765
14109$\frac{2}{7}$

Explanation.—Here we say 7 in 9, 1 and 2 over; in 28, 4 and 0 over; in 7, 1 and 0 over; in 6, 0 times and 6 over; in 65, 9 and 2 over. Beneath this 2 we write the divisor 7 to indicate its division. We may, however, carry on the division by considering the 2 units reduced to tenths, &c., and the quotient becomes 14109·2857.

Thus 2 units, equal to 20 tenths, 7 in 20, 2 and 6 over; 6 tenths are equal to 60 hundredths, 7 in 60, 8 times and 4 over; 4 hundredths are equal to 40 thousandths, 7 in 40, 5 and 5 over; 5 thousandths are equal to 50 tenths of thousandths, &c.

Example 2.—Divide 124789 by 12.

OPERATION.
12)124789
10399$\frac{1}{12}$
or
12)124789
10399·083+

Explanation.—Here again we may either stop at the units and write the remainder 1 over the divisor 12, or we may reduce the 1 unit to tenths, &c., as in the second operation.

Example 3.—Divide £1986 14s. 7½d. by 9.

OPERATION.
9) £1986 14 7½
£220 14 11½

Explanation.—9 in 19, 2 and 1 over; 9 in 18, 2 and 0 over; 9 in 6, 0 and 6 over; £6 are equal to 120s. and 14s. make 134s.; 9 in 134, 14 and 8 over; 8s. are equal to 96d. and 7d. make 103d.; 9 in 103, 11 times and 4 over; 4d. are equal to 16 farthings and 2 farthings make 18 farthings; 9 in 18, 2, i. e., one ninth of 18 farthings is 2 farthings, written thus ½d.

72. In example 3, we are, in reality, required to find one-ninth of the dividend. The obvious meaning is, not that 9 is contained in £1986 14s. 7½d. £220 14s. 11½d. *times*, which would be nonsense, but that £220 14s. 11½d. is the ninth part of £1986 14s. 7½d.: so also in all similar questions.

Notwithstanding this, all such examples are reducible to a species of subtraction. Thus, in the above example, we, for the moment, consider the divisor 9 to be of the same denomination as the dividend, and ascertain how many times £9 will go into (i. e., can be subtracted from) £1986. We get, as a result, 220 times, and a remainder of £6. Then we argue, from the principles already established, that since £9 is contained in £1986 220 times, with a remainder of £6; £220 is contained in £1986 9 times, with a remainder of £6; that is, that the ninth part of £1986 is £220, with a remainder of £6. Next reducing this £6 to shillings, and adding in the 14s., we obtain a total of 184s., and we find that 9s. is contained in 184s. 14 times, with a remainder of 8s., whence we conclude that 14s. is contained in 184s. 9 times, with a remainder of 8s., that is, that the ninth part of 184s. is 14s., with a remainder of 8s., or that the ninth part of £1986 14s. is £220 14s., with 8s. still undivided, &c.

Example 4.—Divide 978964 by 3429.

OPERATION.

$$\begin{array}{r} 3429)978964(285\tfrac{1699}{3429}. \\ 6858 \\ \hline 29816 \\ 27432 \\ \hline 18844 \\ 17145 \\ \hline 1699 \end{array}$$

Explanation.—3429 into 9789 (the smallest number of figures that will contain the divisor) goes 2 times, we therefore put 2 in the quotient. Multiplying 3429 by 2, we get 6858, which we subtract from 9789; and obtain as remainder 2931, which we reduce to the next lower order (tens) and add in the 6 tens, 3429 into 29316 goes 8 times. We therefore place 8 in the quotient. Multiplying 3429 by 8 we get 27432, which we subtract from 29316, and obtain 1884 as a remainder. Reducing to units and adding in the 4, or what amounts to the same thing, bringing down the 4 and writing it after the 1884 we get 18844: and 3429 into 18844 goes 5 times, with a remainder 1699, under which we write the divisor 3429.

73. When the dividend is an abstract number, it is evident that bringing down the next figure and writing it to the right of the remainder, is the same in effect as reducing the remainder to the next lower denomination and adding in the units of that order found in the dividend. Thus, in the last example, bringing down the 6 and writing it directly to the right of the first remainder, 2931, makes the next partial dividend 29316, which is the same as reducing the 2931 to the next lower order and adding to the result the 6 of that order found in the dividend.

Example 5.—Divide 6421284 by 642.

OPERATION.

$$\begin{array}{r} 642)6421284(10002 \\ 642 \\ \hline 1284 \\ 1284 \end{array}$$

Explanation.—642 goes once into 642, and leaves no remainder. Bringing down the next digit of the dividend gives no digit in the quotient, in which, therefore, we put a cipher after the 1. The next digit of the dividend, in the same way, gives no digit in the quotient, in which, consequently, we put another cipher, and, for similar reasons, another in bringing down the next; but the next digit makes the quantity brought down 1284, which contains the divisor twice, and gives no remainder—we put 2 in the quotient.

Note.—*After the first quotient figure is obtained, for each figure of the dividend which is brought down, either a significant figure, or a cipher, must be put in the quotient.*

74. When there is a remainder, we may continue the division, adding decimal places to the quotient, as follows—

EXAMPLE 6. Divide 796347 by 847, and the result by 7234.

OPERATION.

```
847)796347(940·197166, &c.
    7623
    ────
     3404
     3388
     ────
       1670
        847
       ────
        8230
        7623
        ────
         6070
         5929
         ────
          1410
           847
          ────
           5630
           5082
           ────
            5480
            5082
            ────
             398, &c.
7234)940·197166(0·129969, &c.
     723·4
     ─────
     216·79
     144·68
     ──────
      72·117
      65·106
      ──────
       7·0111
       6·5106
       ──────
        ·50056
        ·43404
        ──────
         66526
         65106
         ─────
          1420, &c.
```

75. When the divisor is large, the pupil will find assistance in determining the quotient figure, by finding how many times the first figure of the divisor is contained in the first figure, or, if necessary, the first *two* figures of the dividend. This will give pretty nearly the right figure. Some allowance, must, however, be made for carrying from the product of the other figures of the divisor, to the product of the first into the quotient figure. After multiplying the divisor by the quotient figure, if the product is greater than the corresponding partial dividend, this shows the quotient was taken too great, and must be diminished. If the remainder, after subtraction, is greater than the divisor, the quotient was taken too small, and must be increased.

EXAMPLE 7.—Divide 279 cwt. 3. qrs. 14 lb. 9 oz. by 129.

OPERATION.

```
    cwt. qrs. lb.  oz. cwt. qr.  lb.  oz.  dr.
129)279   3   14   9(   2   0   16   15   9 23/129
    258
    ---
     21
      4=qrs. in cwt.
     --
     87=qrs.
     25=lbs. in qr.
    ---
    449
    174
    ----
    2189=lbs.
    129
    ----
     899
     774
     ---
     125
      16=oz. in lb.
     ---
     759
     125
    ----
    2009=oz.
    129
    ----
     719
     645
     ---
      74
      16=drams in oz.
     ---
     444
     74
    ----
    1184=drams.
    1116
    ----
      23 remainder.
```

EXPLANATION.—129 in 279, i. e., the 129th part of 279 cwt., is 2 cwt., with a remainder of 21 cwt. This 21 cwt. we reduce to quarters by multiplying by 4 and adding in the 3 qrs. The 129th part of 87 qrs. is equal to 0 qr. and we therefore place a 0 in the quarters' place of the quotient. We next reduce qrs. to lbs. by multiplying by 25 and adding in the 14 lbs. of the dividend. We thus obtain 2189 lbs., of which the 129th part is 16 lb., with an undivided remainder of 125 lbs. Reducing 125 lbs. to oz., and adding in the 9 oz., we obtain 2009 oz., of which the 129th part is 15 oz., with an undivided remainder of 74 oz. Reducing the 74 oz. to drams, we obtain 1184 drams, of which the 129th part is 9 drams, with an undivided remainder of 23 drams, under which we place the divisor 129 to indicate its division. Thus we find the total quotient to be 2 cwt. 0 qr. 16 lb. 15 oz. 9$\frac{23}{129}$ drs.

76. The general principles on which the operations in division depend are:—

1st. The quotient arising from the division of the whole dividend by the divisor, is equal to the sum of the quotients arising from the division of the several parts of the dividend by the divisor. (68)

2nd. The divisor and quotient are the factors of the dividend. (65)

3rd. The product of the divisor, by the entire quotient, is equal to the sum of the products of the divisor by the several parts of the quotient. (45)

We ask how many times the divisor is contained in a part of the dividend, and thus a part of the quotient is found; the product of the divisor by this part is taken from the dividend, showing how much of the latter remains undivided; then a part of the remaining dividend is taken and another part of the quotient is found, and the product of the divisor, by it, is taken away from what before remained; and thus the operation proceeds till the *whole* of the dividend is divided, or till the *remainder* is less than the *divisor*.

77. We begin at the left-hand side, because what remains of the higher denomination may still give a quotient in a lower; and the question is, *how often* the divisor will go into the dividend—its different denominations being taken in *any* convenient way. We cannot know how many of the higher we shall have to add to the lower denominations, unless we begin with the higher.

PROOF OF DIVISION.

78. **First Method.**—*Multiply the quotient by the divisor, and to the product add the remainder, if any; the result should be equal to the dividend.* (65)

Example 8.—Divide £5681 13s. 4d. by 700.

```
       £    s.   d.  £   s.  d.               PROOF.
   700)5681 13   4  (8   2   4                £   s.  d.
       5600                                   8   2   4
       ----                                          10
         81                                   ----------
         20                                  81   8   4
       ----                                          10
       1633                                 ------------
       1400                                 811  13   4
       ----                                           7
        233                                ------------
         12                                5681  13   4=£8 2s. 4d. × 700=dividend.
       ----
       2800
       2800
```

Second Method.—*Subtract the remainder, if any, from the dividend; divide the dividend, thus diminished, by the quotient; and if the result is equal to the given divisor, the work is right.*

This is merely doing the same work by a different method.

Third Method.—*Cast the nines out of the divisor and quotient, and multiply the remainders together; add to their product the remainder, if any, after division, and cast the nines out of this sum; the remainder thus obtained should be equal to the remainder obtained by casting the nines out of the dividend.*

Since the divisor and quotient answer to the multiplier and multiplicand, and the dividend to the product, it is evident that the principle of casting out the 9s. will apply to the proof of division as well as to that of multiplication.

Fourth Method.—*Add the remainder and the respective products of the divisor into each quotient figure together; and if the sum is equal to the dividend, the work is right.*

This mode of proof depends upon the principle that the *whole of a quantity is equal to the sum of all its parts.*

Example 9.—Divide 147856 by 97.

```
97)147856(1524
   97*
   ---
    508
    485*
    ---
     235
     194*
     ---
      416
      388*
      ---
       28*
   ------
   147856
```

Note.—The asterisks show the lines to be added.

Exercise 18.

(1)	(2)	(3)	(4)
12)876967	7)891023	9)763457	8)65432·978
$73080\frac{7}{12}$	127289	$84828\frac{5}{9}$	8179·12225

(5)	(6)	(7)	(8)
\$ cts.	\$ cts.	£ s. d.	wks. ds. hrs. min.
9)6789·60	11)4298·76	4)19 6 4	9)69 4 19 30
\$754·40	$\$390·79\frac{7}{11}$	4 16 7	7 5 4 50

9. Divide 798965 by 6428. *Ans.* $124\frac{1893}{6428}$.
10. Divide £176 14s. 6d. by 12. *Ans.* £14 14s. $6\frac{1}{2}$d.
11. Divide 56789 by 741. *Ans.* $76\frac{473}{741}$.
12. Divide 6785158 by 7894. *Ans.* $859\frac{4212}{7894}$.
13. Divide £4728 16s. 2d. by 317. £14 18s. $4\frac{54}{317}$d.
14. Divide \$97896·64 by 429. *Ans.* $\$228·19\frac{313}{429}$.
15. Divide 970763 by 6. *Ans.* 161793·8333+.
16. Divide 71234 by 9. *Ans.* $7914\frac{8}{9}$.
17. Divide 977076 by 47600. *Ans.* $20\frac{25076}{47600}$.
18. Divide 7289 lbs. 6 oz. 4 drs. 2 scr. 13 grs. by 498. *Ans.* 14 lbs. 7 oz. 5 dr. 0 scr. $12\frac{437}{498}$ gr.
19. Divide £157 16s. 7d. by 487. *Ans.* 6s. $5\frac{3}{4}$d. $\frac{59}{487}$.
20. Divide 7867674 by 9712. *Ans.* $810\frac{954}{9712}$.
21. Divide 422 m. 3 fur. 38 rds. by 37. *Ans.* 11 m. 3 fur. 14 rds.

GENERAL PRINCIPLES.

79. If a given divisor is contained in a given dividend a certain number of times, the same divisor will be contained in *double* that dividend *twice* as many times; in *three* times that dividend *thrice* as many times, &c. Hence,

When the divisor remains the same, multiplying the dividend by any number has the effect of multiplying the quotient by the same number.

Thus $9 \div 3 = 3$; 9×2 or $18 \div 3 = 6 = 3 \times 2$, 9×5 or $45 \div 3 = 15 = 3 \times 5$, &c.

80. If a given divisor is contained in a given dividend a certain number of times, the same divisor will be contained in *half* that dividend *half* as many times; in *one-third* of that dividend *one-third* as many times, &c. Hence,

When the divisor remains the same, dividing the dividend by any number has the effect of dividing the quotient by the same number.

Thus $48 \div 3 = 16$; $\frac{48}{2} \div 3$ or $24 \div 3 = 8 = \frac{16}{2}$; $\frac{48}{8} \div 3$ or $6 \div 3 = 2 = \frac{16}{8}$, &c.

81. If a given divisor is contained in a given dividend a certain number of times, *half* that divisor will be contained in the same dividend *twice* as many times, *one-third* of that divisor *thrice* as many times, &c. Hence,

When the dividend remains the same, dividing the divisor by any number has the effect of multiplying the quotient by that number.

Thus $48 \div 6 = 8$; $48 \div \frac{6}{2}$ or $48 \div 3 = 16 = 8 \times 2$; $48 \div \frac{6}{3}$ or $48 \div 2 = 24 = 8 \times 3$, &c.

82. If a given divisor is contained in a given dividend a certain number of times, *twice* that divisor will be contained in the same dividend only *half* as many times, *three* times that divisor only *one-third* as many times, &c. Hence,

When the dividend remains the same, multiplying the divisor by any number has the effect of dividing the quotient by the same number.

Thus $48 \div 2 = 24$; $48 \div$ twice 2 or $48 \div 4 = 12 =$ half of 24.
$48 \div$ eight times 2 or $48 \div 16 = 3 =$ one-eighth of 24, &c.

83. If a given divisor is contained in a given dividend a certain number of times, *twice* that divisor is contained in *twice* that dividend the same number of times; *thrice* that divisor in *thrice* that dividend the same number of times, &c. Hence,

When the divisor and dividend are both multiplied by the same number, the quotient will remain unchanged.

Thus $12 \div 4 = 3$; 24 or twice $12 \div 8$ or twice $4 = 3$; 72 or thrice $24 \div 24$ or thrice $8 = 3$, &c.

84. If a given divisor is contained in a given dividend a certain number of times, *half* that divisor is contained in *half* that dividend the same number of times; *one-third* that divisor in *one-third* that dividend the same number of times, &c. Hence,

When the divisor and dividend are both divided by the same number, the quotient will remain unchanged.

Thus 48÷24=2; 24 or half of 48÷12 or half of 24=2, &c.

TO DIVIDE BY A COMPOSITE NUMBER.

RULE.

85.—*Divide the dividend by one of the factors of the divisor; then the resulting quotient by another factor; and so on till all the factors are used. The last quotient will be the answer.*

Multiply each remainder by all the preceding divisors and add their products to the first remainder, if any, for the true remainder.

When the divisor is separated into only two factors, the rule for finding the true remainder may be thus expressed—

Multiply the last remainder by the first divisor, and to their product add the first remainder, if any; the result will be the true remainder.

EXAMPLE.—Divide 718 lbs. by 72.

OPERATION.

8)718	1st remainder	= 1 lb.	
4)289—1	2nd remainder=3×8	= 9 lb.	
6)59—3	3rd remainder=5×4×3=60 lb.		
9—5	true remainder	70 lb.	*Ans.* 9$\frac{70}{72}$.

That dividing by the factors of a number will give the same quotient as dividing by the number itself, follows directly from Art. 84.

In the last example, dividing by 8 distributes the 718 lbs. into 289 parcels of 8 lbs. each, and leaves a remainder of 1 lb.; dividing next by 4 distributes the 289 parcels into 59 still larger parcels, each containing 4 of the smaller or 8 lb. parcels, and leaves a remainder 3, which is not 3 lbs. but 3 parcels, each of 8 lb.; lastly, dividing the 59 by 6 distributes it into 9 large parcels of 72 lbs. each, and leaves a remainder 5, which is, of course, 5 of the 12 lb. parcels. Hence the reason of the rule for finding the true remainder.

EXERCISE 19.

1. Divide 3766 by 25.	*Ans.* 150$\frac{16}{25}$.
2. Divide 26406 by 42.	*Ans.* 628$\frac{30}{42}$.
3. Divide 25431 by 96.	*Ans.* 264$\frac{87}{96}$.
4. Divide £24 17s. 6d. by 24.	*Ans.* £1 0s. 8$\frac{3}{4}$d.
5. Divide £740 13s. 4d. by 49.	*Ans.* £15 2s. 3d$\frac{3}{4}$.$\frac{1}{49}$.
6. Divide £547 12s. 4d. by 56.	*Ans.* £9 15s. 6d$\frac{3}{4}$.$\frac{40}{56}$.
7. Divide 6789436 by 35.	*Ans.* 193983$\frac{31}{35}$.
8. Divide 753293 by 147 (=7×7×3).	*Ans.* 5124$\frac{65}{147}$.
9. Divide 1798 lbs. 6 oz. 11 dwt. 9 grs. by 81.	*Ans.* 22 lbs. 2 oz. 9 dwt. 0$\frac{57}{81}$ grs.

86. When both the divisor and the dividend are denominate numbers—

RULE.

Reduce both the divisor and the dividend to the lowest denomination contained in either, and then proceed as in Art. 71.

Example 1.—Divide £37 5s. 9¼d. by 3s. 6½d.

```
 s. d.              £  s. d.
 3 6½               37 5  9¼
12                  20
--                  ---
42                  745
 4                   12
---                ----
170 farthings.     8949
                      4
                  -----
            170)35797(210 97/170 times.
                 340
                 ---
                 179
                 170
                 ---
                  97
```

87. In the above and all similar questions we are required to find what fraction the divisor is of the dividend; or, in other words, how often the divisor is contained in, or can be subtracted from, the dividend, and the quotient must necessarily be an *abstract* number.

Example 2.—Divide 729 cwt. 3 qrs. 16 lb. by 3 qrs. 9 lb. 7 oz.

```
qrs. lbs. oz.          cwt. qrs. lbs.
  3   9   7             729   3   16
 25                       4
 --                     ----
 84                     2919
 16                       25
---                    -----
511                    14611
84                     5838
----                   -----
1351 oz.               72991
                          16
                      ------
                      437946
                      72991
                     -------
              1351)1167856 oz. (864 592/1351 times.
                   10808
                   -----
                    8705
                    8106
                    ----
                    5996
                    5404
                    ----
                     592
```

Exercise 20.

1. Divide £8968 13s. 7½d. by £491 12s. 0¼d. *Ans.* $18\frac{115068}{471937}$.
2. Divide 1027 m. 1 fur. 6rds. by 17 m. 5 fur. 27 rds. *Ans.* 58.
3. Divide £171 1s. 10½d. by £57 0s. 7½d. *Ans.* 3.
4. Divide 9 lb. 9 oz. 3 dwts. 12 grs. by 5 dwts. 9 grs. *Ans.* 436.
5. Divide 2366 acres 3 roods 36 rds. by 91 acres 6 rds. *Ans.* 26.

---

88. When the dividend alone contains decimal places, the preceding rules are sufficient; but when the divisor contains decimals, it becomes necessary to prepare the quantities for the division according to the following—

RULE.

Remove the decimal point as many places to the right in both the dividend and the divisor, as there are decimals in the divisor, and then proceed as in Art. 71.

This is simply multiplying both dividend and divisor by the same number, and therefore (Art. 83) does not affect the quotient. Thus removing the decimal point one place to the right, in both dividend and divisor, is equivalent to multiplying each by 10; two places, the same as multiplying each by 100; three places, by 1000, &c.

Example 1.—Divide 87·6 by ·0009.

Multiplying each by 10000, or, in other words, removing the decimal point four places to the right, in each, (since there are *four* decimals in the divisor,) gives us 876000÷9, and this (Art. 83) must give the same quotient as 87·6÷·0009, therefore

87·6÷·0009=876000÷9=97333·33, &c.

Example 2.—Divide ·06 by 8·934.

·06÷8·934=60÷8934.

8934)60·000(0·0067, &c.
53·604

6·3960
6·2538

1422

Removing the decimal point *three* places to the right, in each, we get 60÷8934, and we then proceed thus: 8934 into 60 (units), 0 (units) times; set down 0 with the decimal point after it; 8934 into 600 (tenths), 0 times; into 6000 (hundredths) 0 times; into 60000 (thousandths), 6 (thousandths) times, &c.

Example 3.—Prepare 93·004÷·0000069 for division.

Ans. 93·004÷·0000069=930040000÷69.

Exercise 21.

1. 43÷·0006947=430000000÷6947.
2. 9378·92÷9·7891=93789200÷97891.
3. 4·96723÷23·934=4967·23÷23934.
4. ·793÷·49=79·3÷49.

5. ·001÷674·937=1÷674937.
6. Divide 47·655 by 4·5. *Ans.* 10·59.
7. Divide 756·98 by 76·78612. *Ans.* 9·864+.
8. Divide 47·5782975 by 26·175. *Ans.* 1·8177.
9. Divide 1 by 7·6845. *Ans.* 0·1309+.
10. Divide 75·347 by 0·3829. *Ans.* 196·7798+.
11. Divide ·0002 by ·000000008. *Ans.* 25000.

CONTRACTIONS IN DIVISION.

89. To divide by 10, 100, 1000, &c.

Remove the decimal point as many places to the left in the dividend as there are 0s in the divisor.

90. To divide by 25.

Multiply by 4, and divide by 100.

REASON.—$25=\frac{100}{4}$.

91. To divide by 15, 35, 45, or 55.

Double the dividend, and divide the product by 30, 70, 90, *or* 110, *as the case may be.*

REASON.—This method is simply doubling both the divisor and dividend. We must therefore divide the remainder, if any, by 2, for the *true* remainder.

92. To divide by 125.

Multiply the dividend by 8, *and divide the product by* 1000.

REASON.—This contraction is multiplying both the dividend and divisor by 8. For the *true* remainder, therefore, we must divide the remainder, if any, by 8.

93. To divide by 75, 175, 225, or 275.

Multiply the dividend by 4, *and divide the product by* 300, 700, 900, *or* 1100, *as the case may be.*

REASON.—$75=\frac{300}{4}$, $175=\frac{700}{4}$, &c. For the *true* remainder, divide the remainder, if any thus found, by 4.

94. When there are many decimals in the dividend and but few are required in the quotient, we may abbreviate the division by the following—

RULE.

Proceed as in Art. 71 *till the decimal point is placed in the quotient, and then cut off a digit to the right hand of the divisor, at each new digit of the quotient; remembering to carry what would have been obtained by the multiplication of the digit neglected—unity if this multiplication would have produced more than* 5 *and less than* 15; 2 *if more than* 15 *and less than* 25, &c.

Example.—Divide 754·337385 by 61·347.

```
      Ordinary Method.              Contracted Method.
61347)754337·385(12·296      61347)754337·385(12·296
      61347                        61347
      -----                        -------
      14086|7·                     140867·
      12269|4·                     122694·
      -----|--                     -------
       1817|3·3                     18173·
       1226|9·4                     12269·
       ----|---                     ------
        590|3·98                     5904·
        552|1·23                     5521·
        ---|----                     -----
         38|2·755                     383·
         36|8·082                     368·
         --|-----                     ----
          1|4·6730                     15·
```

According as the denominations of the quotient become small, their products by the lower denomination of the divisor become inconsiderable, and may be neglected, and consequently, the portions of the dividend from which they would have been subtracted. What should have been *carried* from the multiplication of the digit neglected—since it belongs to a higher denomination than what is neglected—must still be retained.

Exercise 22.

1. The Ontario, Simcoe, and Huron Railway is 95 miles in length, and cost $3300000. What was the cost per mile?

2. The Rideau Canal is 126 miles in length, and cost $3860000. What was the average cost per mile?

3. The distance of the earth from the sun is 95270400 miles. How long would it take a cannon ball, going at the rate of 28800 miles per day, to reach the sun?

4. The national debt of France is 1145012096 dollars, and the number of inhabitants is 35781628. What is the amount of indebtedness of each individual?

5. The national debt of Great Britain is 3764112127 dollars, and the number of inhabitants is 27475271. What is the amount of indebtedness of each individual?

6. What is the ninth part of $972?

7. What is each man's part, if $972 be divided equally among 108 men?

8. Divide a legacy of $8526 equally between 294 persons.

9. Divide 340480 ounces of bread equally between 792 persons.

10. A cubic foot of distilled water weighs 1000 ounces. What will be the weight of one cubic inch?

11. How many Sabbath days' journeys (each 1155 yards) in the Jewish day's journey, which was equal to 33 miles and 2 furlongs English?

12. How many pounds of butter, 19 cents per lb., would purchase a cow, the price of which is $47·50?

13. Divide 978·634 by 96·34762.

14. Divide 729 bush. 1 pk. 1 gal. 1 qt. 1 pt. by 297.

15. Divide 179 cwt. 3 qr. 4 lb. 16 oz. by 9 lb. 7 oz. 8 drs.

16. The circumference of the earth is about 25000 miles; if a vessel sails 93 m. 4 fur. 7 rds. a day, how long will it require to sail round the earth?

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the articles of the section.*

1. What is division? (58)
2. What is the divisor? (59)
3. What is the dividend? (60)
4. What is the quotient? What is the derivation of the word "quotient?" (61)
5. Explain when the quotient will be equal to unity, and when greater or less than unity. (62)
6. Under what circumstances does a remainder arise in division? (63)
7. What is the denomination of the remainder? (63)
8. Why can it never be as great as the divisor? (63)
9. What is the correspondence between the minuend and the subtrahend in subtraction and the divisor and the dividend in division? (64)
10. What may we consider as the factors of the dividend? (65)
11. How many ways have we of expressing the division of one quantity by another? What are they? (66)
12. When a quantity consisting of several terms, connected by the sign ×, is to be divided by any number, how may the work be performed? (67)
13. When a quantity consisting of several terms, connected by the signs + or −, contained within brackets, is to be divided, what must be done upon removing the brackets? (68)
14. Give the general rule for division. (71)
15. In the question "Divide 11 m. 7 fur. 20 per. 3 yrds. by 279," explain what is really required. (72) Show that all such questions are reducible to a species of subtraction. (72)
16. In dividing abstract numbers, explain what bringing down the next figure of the dividend is equivalent to. (73)
17. When there is a remainder, how is it to be written? (71, Example 1)
18. What are the three general principles upon which the operations of division depend? (76)
19. Why do we begin dividing at the left-hand side? (77)
20. How may division be proved? (78)
21. The divisor remaining unchanged, what effect has multiplying the dividend by any number? (79)
22. The divisor remaining unchanged, what effect has dividing the dividend by any number? (80)
23. The dividend remaining unchanged, what effect has dividing the divisor by any number? (81)
24. The dividend remaining unchanged, what effect has multiplying the divisor by any number? (82)
25. What is the effect upon the quotient when the divisor and the dividend are both multiplied by the same number? (83)
26. What is the effect upon the quotient when the divisor and the dividend are both divided by the same number? (84)
27. How do we divide by a composite number? (85)
28. When we divide by the divisors of a composite divisor, how do we obtain the correct remainder? (85)
29. When the divisor is separated into only two factors, how may the rule for obtaining the correct remainder be worded? (85)
30. When the divisor and the dividend are both denominate numbers, what is the rule? (86)
31. When one denominate number is divided by another, what kind of a number must the quotient always be? (87)

32. In the question "Divide 37 lb. 2 oz. 15 dr. by 1 lb. 9 oz. 11 dr.," what are we in reality required to do? (87)
33. When the divisor contains decimals, how do we proceed? (88) Upon what principle do we do this? (88)
34. How do we divide by 1, followed by any number of 0s? (89)
35. How do we contract the work when dividing by 25? How by 15, 35, 45, or 55? (90, 91)
36. How do we divide by 125? How by 75, 175, 225, or 275? (92, 93)
37. How do we abbreviate the work when there are many decimals in the dividend and but few are required in the quotient? (94)

EXERCISE 23.

MISCELLANEOUS EXERCISE.

(*On preceding rules.*)

1. Multiply 789643 by 999998.

2. Read the following numbers: 67813420·021030046, 72000000·000000072, 1001000100·0010000010000001.

3. Express 709, 4376, 9999, 86004, and 3947596 in Roman numerals.

4. Multiply 749 lb. 10 oz. avoirdupois by 72.

5. What is the price of 17 pairs of gloves at 4s. 7½d. per pair?

6. The planet Neptune is 2850 millions of miles from the sun. How long would it take a locomotive to travel from the sun to Neptune, at the rate of 30 miles an hour?

7. Reduce £729 17s. 6½d. to dollars and cents.

8. From $10000 subtract $9876·23.

9. Write down five hundred and twenty billions, six millions, two thousand and forty-three, and five thousand and sixteen trillionths.

10. Reduce 7964327 inches to acres, roods, &c.

11. Add together the following quantities: $729·43, $16·70, $976·81, $9987·17, $429·00, $129·19.

12. Multiply 6 weeks 4 days 3 hours 17 minutes by 429.

13. Take the number 741, and, by removing the decimal point: (1) multiply it by 1000000; (2) divide it by 100000; (3) make it millions; (4) make it billionths; (5) make it trillionths; (6) make it hundredths of thousandths; (7) make it tenths.

14. Multiply 78·96 by ·00042.

15. How many hogsheads of sugar, each containing 13 cwt. 2 qrs. 14 lbs., may be put on board a ship of 324 tons burden?

16. A farmer's yearly income was 9237 dollars. He paid for repairing his house 136 dollars, for hired help on his farm 4 times as much lacking 95 dollars, and for other expenses 1902 dollars. How much does he save yearly?

17. How many suits of clothes can be made from a piece of cloth containing 39 yrds. 2 qrs. 3 nls.; each suit requiring 3 yrds. 1 qr. 2 nls.?

18. There is a farm consisting of 732 acres; 25 acres of which is

planted with corn and potatoes; 197 acres sown with rye; 156 with oats; 97 with wheat; 199 is pastured; and the remainder is meadow. How many acres of meadow?

19. Bought 96 acres 3 roods 17 perches of land, for which I pay $7764; what did I pay for it per perch?

20. A lady, having 312 dollars, paid for a bonnet 20 dollars, for a shawl 75 dollars, for a silk dress 97 dollars, and for some delaines 83 dollars; how much had she remaining?

21. A silversmith received 36lb. 8 oz. 14 dwt. 16 grs. of silver to make 12 tankards: what would the weight of each tankard be?

22. I bought four fields; in the first there were 6 acres 3 rds. 12 perches; in the second, 7 acres 2 roods; in the third, 9 acres and 13 perches; in the fourth, 5 acres 2 roods 36 perches. How much in all.

23. A merchant expended 294 dollars for broadcloth, consisting of three different kinds; the first at 5 dollars a yard; the second at 7 dollars; and the third at 9 dollars a yard. He had as many yards of one kind as of another—how many yards of each kind did he buy?

24. A silversmith made three dozen spoons, weighing 5 lb. 9 oz. 8 dwt.; a tea-pot weighing 3 lb. 2 oz. 16 dwt. 16 grs.; two pair of silver candlesticks, weighing 4 lb. 6 oz. 17 dwt.; a dozen silver forks, weighing 1 lb. 8 oz, 19 dwt. 22 grs.; what was the weight of all the articles?

25. Reduce £972 11s. 11¼d. to dollars and cents.

26. Reduce 179 lbs. 3 oz. 3 dr. 1 scr. 14 grs. to grains.

27. There is a house 56 feet long, and each of the two sides of the roof is 25 feet wide; how many shingles will it take to cover it, if it require 6 shingles to cover a square foot?

28. A merchant bought 4 bales of cotton; the first contained 6 cwt. 2 qr. 11 lb.; the second, 5 cwt. 3 qr. 16 lb.; the third, 8 cwt. 0 qr. 7 lb.; the fourth, 3 cwt. 1 qr. 17 lb. He sold the whole at 15 cents a pound; what did it amount to?

29. A merchant has 29 bales of cotton cloth, each bale containing 57 yards; what is the value of the whole at 15 cents a yard?

30. A man willed an estate of $370129 to his two children and wife, as follows: to his son, $139468; to his daughter, $98579; and to his wife the remainder. How much did he will to his wife?

31. Divide £1694 16s. 0¼½d. by £9 19s. 11¾d.

32. Reduce £19 19s. 11¾d. to dollars and cents.

33. A merchant having purchased 12 cwt. of sugar, sold at one time 3 cwt. 2 qrs. 11 lb., and at another time he sold 4 cwt. 1 qr. 15 lb.; what is the remainder worth, at 15 cents per pound?

34. Bought 4 chests of hyson tea; the weight of the first was 2 cwt. 0 qr. 17 lb.; the second, 3 cwt. 2 qrs. 15 lb.; the third, 2 cwt. 1 qr. 20 lb.; the fourth, 5 cwt. 3 qr. 17 lb.; what is the value of the whole at 37½ cents a pound?

35. Express 100200300709 in Roman numerals.

36. Divide 48·2 by 76·8437.

37. Divide 128·4 by ·000000066.

38. From $2789·27 take 17 times $63·29.

39. Add together $278·48, $417·16, $11·27, $2110·40, $728·15, and £29 6s. 11¾d. and divide the sum by 178.

40. In 1857 the total number of volumes in the Common School and other Public Libraries of Canada West was estimated at 491544 and the number of libraries at 2076. How many volumes were there upon an average to each library?

SECTION III.

PROPERTIES OF NUMBERS, PRIME NUMBERS, MEASURES, GREATEST COMMON MEASURE, LEAST COMMON MULTIPLE, SCALES OF NOTATION, AND APPLICATION OF THE FUNDAMENTAL RULES TO DIFFERENT SCALES. DUODECIMALS.

1. A divisor, or measure of a number, is a number which will divide it exactly; that is leaving no remainder.

2. A multiple of a number is a number of which the given number is a divisor.

3. An integer, or integral number, is a whole number.

4. Integers are either *prime* or *composite*, *odd* or *even*.

5. An Even Number is that of which 2 is a divisor.

6. An odd number is that of which 2 is not a divisor.

7. A Prime Number is one which has no integral divisor except unity and itself, thus 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, &c., are primes.

8. A Composite Number is a number which is not prime; or is a number which has other *integral* divisors besides unity and itself, thus 4, 6, 9, 10, 12, 14, 15, 16, 21, &c., are composite numbers.

9. The Factors of a number are those numbers which, when multiplied together, *produce* or *make* it.

10. Factors are sometimes called measures, submultiples, or aliquot parts.

11. A Common Measure of two or more numbers, is a number which will divide each of them without a remainder; thus 7 is a common measure of 14, 35, and 63.

12. Two or more numbers are prime to one another when they have no common divisor except unity; thus, 9 and 14 are "prime to each other."

Hence all prime numbers are prime to each other; but composite numbers may or may not be prime to one another.

13. Commensurable Numbers are those which have some common divisor.

Thus 55 and 33 are commensurable, the common divisor being 11.

14. Incommensurable Numbers are those which are prime to one another.

Thus 55 and 34 are incommensurable.

15. A Square Number is one which is composed of two equal factors.

Thus $25=5\times5$ is a square number; so also $64=8\times8$, &c.

16. A Cube Number is one which is composed of three equal factors.

Thus $343=7\times7\times7$ is a cube number; so also $27=3\times3\times3$, &c.

17. A perfect Number is one which is exactly equal to the sum of all its divisors.

Thus $6=1+2+3$ is a perfect number; so also $28=1+2+4+7+14$ is a perfect number.

All the numbers known to which this property really belongs, are the eight following: 6; 28; 496; 8128; 33550336; 8589869056; 137438691328; and 2305843008139952128.

NOTE.—All perfect numbers terminate with 6, or 28.

18. Amicable Numbers are such pairs of integers that each of them is exactly equal to the sum of all the divisors of the other.

Thus 220 and 284 are amicable; for, $220=1+2+4+71+142$, which are all the divisors of 284, and $284=1+2+5+11+4+10+22+20+44+55+110$, which are all divisors of 220.

Other amicable numbers are 17296 and 18416; also 9363583 and 9437056.

19. By the term *properties of numbers*, is meant those qualities or elements which are inseparable from them. Some of the most important properties of numbers are the following—

I. The sum of two or more even numbers is an even number.

II. The difference of two even numbers is an even number.

III. The sum or difference of two odd numbers is an even number.

IV. The sum of three, five, seven, &c., odd numbers, is an odd number.

V. The sum of two, four, six, eight, &c., odd numbers, is an even number.

VI. The sum or difference of an even and an odd number, is an odd number.

VII. The product of two even numbers, or of an even and an odd number, is an even number.

VIII. If an even number be divisible by an odd number, the quotient will be an even number.

IX. The product of any number of factors will be even if one of the factors be even.

X. An odd number is not divisible by any even number.

XI. The product of any number of factors is odd if they are all odd.

XII. If an odd number divide an even number, it will also divide half of it.

XIII. Any number that measures two others must likewise measure their *sum*, their *difference*, and their *product*.

Thus, if 6 goes into 24 four times, and into 18 three times, it will go into 24+18 or 42, three plus four, or seven times.

Also, if 6 goes into 24 four times, and into 42 seven times, it will go into 42−24 or 18, seven minus four, or three times.

Lastly, if 6 goes into 24 four times, and into 12 twice, it will evidently go into 12 times 24, twelve times 4 times, or 48 times.

XIV. If one number measure another, it must likewise measure any multiple of that other.

Thus, if 7 measures 21, it must evidently measure 6 times 21, or 11 times 21, or 17 times 21, &c.

XV. Any number, expressed by the decimal notation, divided by 9, will leave the same remainder as the sum of its digits divided by 9. (See Art. 55, Sec. II.)

This property of the number 9 affords an ingenious method of proving each of the fundamental rules. The same property belongs to the number 3; for 3 is a measure of 9, and will therefore be contained an exact number of times in any number of 9s. But it belongs to no other digit.

The preceding is not a *necessary* but an *incidental* property of the number 9. It arises from the *law of increase* in the decimal notation. If the *radix* of the system were 8, it would belong to 7; if the radix were 12, it would belong to 11; and, universally, it belongs to the number that is *one less* than the *radix* of the system of notation.

XVI. If the number 9 be multiplied by any *single* digit, the *sum* of the figures composing the product will make 9.

Thus 9 × 4=36, and 3+6=9; so also 8 × 9=72 and 7+2=9.

XVII. If we take any two numbers whatever; then *one* of them, or their *sum*, or their *difference*, is divisible by 3.

Thus, take 11 and 17; though neither the numbers themselves, nor their sum, is divisible by 3, yet their difference is, for it is 6.

XVIII. Any number divided by 11, will leave the *same remainder* as the sum of its *alternate* digits in the *even* places, reckoning from the right, taken from the sum of its alternate digits in the *odd* places, increased by 11, if necessary.

Take any number as 38405603, and mark the alternate figures. Now the sum of those marked, viz: 8+0+6+3=17. The sum of the others, viz: 3+4+5+0=12. And 17−12=5, the remainder sought. That is, 38405603 divided by 11, will leave 5 remainder.

Again, take 5847362, the sum of the marked figures is 14; the sum of those not marked is 21. Now 21 taken from 25, (*i. e.* 14 increased by 11) leaves 4, the remainder sought=remainder obtained by dividing 5847362 by 11.

XIX. Any number ending in 0, or an even number, is divisible by 2.

XX. Any number ending in 5 or 0 is divisible by 5.

XXI. Any number ending in 0 is divisible by 10.

XXII. When two right-hand figures are divisible by 4, the whole is divisible by 4.

XXIII. When the three right-hand figures are divisible by 8, the whole number is divisible by 8.

XXIV. When the sum of the digits of a number is divisible by 9, the number itself is divisible by 9.

XXV. When the sum of the digits of a number is divisible by 3, the number itself is divisible by 3.

XXVI. When the sum of the digits, standing in the *even* places, is equal to the sum of the digits standing in the *odd* places, the number is divisible by 11.

Thus to illustrate the last five properties.

The number 7416 is divisible by 4, because 16, the last two digits, is divisible by 4.
—— is divisible by 8, because 416, its last three digits, is divisible by 8.
—— is divisible by 9, because the sum of its digits, 7+4+1+6=18, is divisible by 9.
—— is divisible by 3, because the sum of its digits, 7+4+1+6=18, is divisible by 3.

So also the number 4567321 is divisible by 11, since the sum of the digits in the odd places, 1+3+6+4=14=2+7+5, the sum of the digits in the even places.

XXVII. Every *composite number* may be resolved into *prime factors*.

For since a composite number is produced by multiplying two or more factors together, it may evidently be resolved into those factors; and if these factors themselves are *composite*, they also may be resolved into *other* factors, and thus the analysis may be continued until all the factors are *prime* numbers.

XXVIII. The *least* divisor of any number is a prime number.

For every whole number is either prime or composite (Art. 4); but a composite number can be resolved into factors (XXVII): consequently, the *least* divisor of any number must be a prime number.

XXIX. Every prime number, except 2, if increased or diminished by 1 is divisible by 4. (See table of prime numbers on next page.)

XXX. Every prime number except 2, is odd; and therefore terminates in an odd digit.

NOTE.—It must not be inferred from this that all *odd* numbers are *prime*.

XXXI. All prime numbers, except 2 and 5, must terminate with 1, 3, 7, or 9. Every number that ends in any other digit than 1, 3, 7, or 9, is a composite number.

For all prime numbers, except 2, must end in an odd digit (XXX), and all numbers ending in 5 are divisible by 5.

XXXII. Every prime number, except 2 and 3, if increased or diminished by 1, is divisible by 6.

20. To find the prime numbers between any given limits.

RULE.

Write down all the odd numbers, 1, 3, 5, 7, 9, &c. Over every third from 3 write 3; over every fifth from 5 write 5; over every seventh from 7 write 7; over every eleventh from 11 write 11; and so on.

Then all the numbers which are thus marked are composite; and the others, together with 2, are prime.

Also the figures thus placed over, are the factors of the numbers over which they stand.

EXAMPLE.

Find all the prime numbers less than 100.

				3			3·5	
1	3	5	7	9	11	13	15	17
	3·7		5	3			3·11	5·7
19	21	23	25	27	29	31	33	35
	3·13			3·5		7	3·17	
37	39	41	43	45	47	49	51	53
5·11	3·19			3·7	5·13		3·23	
55	57	59	61	63	65	67	69	71
	3·5	7·11		3		5·17	3·29	
73	75	77	79	81	83	85	87	89
7·13	3·31	5·19		3·11				
91	93	95	97	99				

Hence, rejecting all the numbers which have *superiors*, the primes less than 100 are 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, together with the number 2.

This process may be extended indefinitely, and is the method by which primes are found even by modern computators. It was invented by Eratosthenes, a learned librarian at Alexandria (Born B. C. 275). He inscribed the series of odd numbers upon parchment, then cutting out such numbers as he found to be composite, his parchment with its holes somewhat resembled a *sieve;* hence, this method is called "*Eratosthenes' Sieve.*"

TABLE OF PRIME NUMBERS FROM 1 TO 3407.

1	173	409	659	941	1223	1511	1811	2129	2423	2741	3079
2	179	419	661	947	1229	1523	1823	2131	2437	2749	3083
3	181	421	673	953	1231	1531	1831	2137	2441	2753	3089
5	191	431	677	967	1237	1543	1847	2141	2447	2767	3109
7	193	433	683	971	1249	1549	1861	2143	2459	2777	3119
11	197	439	691	977	1259	1553	1867	2153	2467	2789	3121
13	199	443	701	983	1277	1559	1871	2161	2473	2791	3137
17	211	449	709	991	1279	1567	1873	2179	2477	2797	3163
19	223	457	719	997	1283	1571	1877	2203	2503	2801	3167
23	227	461	727	1009	1289	1579	1879	2207	2521	2803	3169
29	229	463	733	1013	1291	1583	1889	2213	2531	2819	3181
31	233	467	739	1019	1297	1597	1901	2221	2539	2833	3187
37	239	479	743	1021	1301	1601	1907	2237	2543	2837	3191
41	241	487	751	1031	1303	1607	1913	2239	2549	2843	3203
43	251	491	757	1033	1307	1609	1931	2243	2551	2851	3209
47	257	499	761	1039	1319	1613	1933	2251	2557	2857	3217
53	263	503	769	1049	1321	1619	1949	2267	2579	2861	3221
59	269	509	773	1051	1327	1621	1951	2269	2591	2879	3229
61	271	521	787	1061	1361	1627	1973	2273	2593	2887	3251
67	277	523	797	1063	1367	1637	1979	2281	2609	2897	3253
71	281	541	809	1069	1373	1657	1987	2287	2617	2903	3257
73	283	547	811	1087	1381	1663	1993	2293	2621	2909	3259
79	293	557	821	1091	1399	1667	1997	2297	2633	2917	3271
83	307	563	823	1093	1409	1669	1999	2309	2647	2927	3299
89	311	569	827	1097	1423	1693	2003	2311	2657	2939	3301
97	313	571	829	1103	1427	1697	2011	2333	2659	2953	3307
101	317	577	839	1109	1429	1699	2017	2339	2663	2957	3313
103	331	587	853	1117	1433	1709	2027	2341	2671	2963	3319
107	337	593	857	1123	1439	1721	2029	2347	2677	2969	3323
109	347	599	859	1129	1447	1723	2039	2351	2683	2971	3329
113	349	601	863	1151	1451	1733	2053	2357	2687	2999	3331
127	353	607	877	1153	1453	1741	2063	2371	2689	3001	3343
131	359	613	881	1163	1459	1747	2069	2377	2693	3011	3347
137	367	617	883	1171	1471	1753	2081	2381	2699	3019	3359
139	373	619	887	1181	1481	1759	2083	2383	2707	3023	3361
149	379	631	907	1187	1483	1777	2087	2389	2711	3037	3371
151	383	641	911	1193	1487	1783	2089	2393	2713	3041	3373
157	389	643	919	1201	1489	1787	2099	2399	2719	3049	3389
163	397	647	929	1213	1493	1789	2111	2411	2729	3061	3391
167	401	653	937	1217	1499	1801	2113	2417	2731	3067	3407

When it is required to determine whether a given number is a prime, we first notice the terminating figure; if it is different from 1, 3, 7, or 9, the number is composite; but if it terminate with one of the above digits, we must endeavour to divide it with some one of the primes, as found in the table, commencing with 3. There is no necessity for trying 2, for 2 will divide only the even numbers. If we proceed to try all the successive primes of the table until we reach a prime which is not less than the square-root of the number, without finding a divisor, we may conclude with certainty that the number is a *prime.*

The reason why we need not try any primes greater than the square-root of the number, is drawn from the following consideration: If a composite number is resolved into two factors, one of which is less than the square-root of the number, the other must be greater than the square-root.

The square of the last prime given in our table is 11607649; hence, this table is sufficiently extended to enable us to determine whether any number not exceeding 11607649 is a prime. It is obvious that numbers may be proposed which would require by this method very great labor to determine whether they are primes, still this is the only sure and general method as yet discovered.

21. To Resolve a Composite Number into its Prime Factors.

RULE.

Divide the given number by the smallest number which will divide it without a remainder; then divide the quotient in the same way, and thus continue the operation till a quotient is obtained which can be divided by no number greater than 1. *The several divisors with the last quotient, will be the prime factors required.* (19–*XXVII.*)

Reason.—Every *division* of a number, it is plain, resolves it into *two factors*, viz.: the divisor and the quotient. But according to the rule, the divisors, in every case, are the *smallest* numbers that will divide the given number or the successive quotients without a remainder; consequently they are all *prime* numbers. (19-XXVIII.) And since the division is continued till a quotient is obtained, which cannot be divided by any number but unity or itself, it follows that the *last* quotient must also be a *prime* number; for, a prime number is one which cannot be exactly divided by any whole number except *unity* and *itself*. (Art. 7.)

Note.—Since the *least divisor* of every number is a *prime* number, it is evident that a composite number may be resolved into its prime factors by dividing it continually by *any prime number* that will divide the given number and the successive quotients without a remainder. Hence,

A composite number can be divided by any of its *prime factors* without a remainder, and by the product of any two or more of them, but by *no other number*.

Thus the prime factors of 42 are 2, 3, and 7. Now 42 can be divided by 2, 3, and 7; also by 2 × 3, 2 × 7, 3 × 7, and 2 × 3 × 7; but it can be divided by no other number.

Example 1.—Resolve 210 into its prime factors.

OPERATION.
2)210
3)105
5)35
7

We first divide the given number by 2, which is the least number that will divide it without a remainder, and which is also a prime number. We next divide by 3, then by 5. The several divisors and the last quotient are the prime factors required.

Ans. 2, 3, 5, and 7.

PROOF.
$2 \times 3 \times 5 \times 7 = 210$

Example 2.—Resolve 728 into its prime factors.

OPERATION.
2)728
2)364
2)182
7)91
13

Therefore, $2 \times 2 \times 2 \times 7 \times 13$, or $2^3 \times 7 \times 13$, are the prime factors of 728.

EXERCISE 24.

1. Resolve 11368 into its prime factors. *Ans.*[*] $2^3 \times 7^2 \times 29$.
2. What are the prime factors of 2934? *Ans.* $2 \times 3^2 \times 163$.
3. What are the prime factors of 1011? *Ans.* 3×337.
4. What are the prime factors of 1000? *Ans.* $2^3 \times 5^3$.
5. What are the prime factors of 1024? *Ans.* 2^{10}.
6. What are the prime factors of 32320? *Ans.* $2^6 \times 5 \times 101$.
7. What are the prime factors of 707? *Ans.* 7×101.
8. What are the prime factors of 1118? *Ans.* $2 \times 13 \times 43$.

DIVISORS.

22. From Art. 21, Note, for finding all the divisors of any number, we deduce the following—

RULE.

Resolve the number into its prime factors; form as many series of terms as there are prime factors, by making 1 *the first term of each series, the first power of one of the prime factors for the second term, the second power of this factor for the third term, and so on until we reach the highest that occurred in the decomposition. Then multiply these series together, and the partial products thus obtained will be the divisors sought.*

EXAMPLE 1.—What are the divisors of 48?

Here we find $48 = 2^4 \times 3$. Therefore our series of terms will be 1··2··4··8··16 and 1··3; multiplying these together.

1··2··4··8··16
1··3

1··2··4··8··16··3··6··12··24··48

Therefore the divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48.

We begin each series with 1, because, were we not to do so, the different powers of the prime factors would not themselves appear among the partial products.

EXAMPLE 2.—What are the divisors of 360.

The prime factors of 360 are $2^3 \times 3^2 \times 5$ and therefore the series are 1··2··4··8; 1··3··9 and 1··5.

OPERATION.

1··2··4··8
1··3··9

1··2··4··8··3··6··12··24··9··18··36··72=products of 1st and 2nd series
1··5

1··2··4··8··3··6··12··24··9··18··36··72··5··10··20··40··15··30··60··120··45··90··180··360.

Therefore the divisors of 360 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360.

---

[*] The small figures written to the right of the factors and above the line are called exponents, and show how often the digit is taken as factor.

1. What are the divisors of 100? *Ans.* 1, 2, 4, 5, 10, 20, 25, 50, 100.
2. What are the divisors of 810?
Ans. 1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 81, 90, 135, 162, 270, 405, 810.
3. What are the divisors of 920?
Ans. 1, 2, 4, 5, 8, 10, 20, 23, 40, 46, 92, 115, 184, 230, 460, 920.
4. What are the divisors of 25000?
Ans. 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, 625, 1000, 1250, 2500, 3125, 5000, 6250, 12500, 25000.

NUMBER OF DIVISORS.

23. Since the series of terms which we multiplied together, by the last rule, to obtain the divisors of any number commenced with 1, it follows that the number of terms in each series will be one more than the units in the exponent of the factors used.

Hence, to find the *number* of divisors of any number without actually setting them down, we have the following—

RULE.

Resolve the number into its prime factors and express them as in examples 3, 4, and 6, in Art. 21. Increase each exponent by unity and multiply the resulting numbers together. The product will be the number of divisors.

Example.—How many divisors has 4320?

$4320=2^5\times3^3\times5$. Here the exponents are 5, 3, and 1: each of which being increased by one, we obtain 6, 4, and 2, the continued product of which is $6\times4\times2=48$=the number of divisors sought.

Exercise 26.

1. How many divisors has 88200?	*Ans.* 108.
2. How many divisors has 3500?	*Ans.* 24.
3. How many divisors has 6336?	*Ans.* 42.
4. How many divisors has 824?	*Ans.* 8.
5. How many divisors has 49000?	*Ans.* 48.
6. How many divisors has 81000?	*Ans.* 80.
7. How many divisors has 75600?	*Ans.* 120.
8. How many divisors has 25600?	*Ans.* 33.

GREATEST COMMON MEASURE.

24. The greatest common measure, or greatest common divisor of two or more numbers, is the greatest number that will divide each of them without a remainder.

25. To find a common divisor or common measure of two or more numbers—

RULE.

Resolve the given numbers into their prime factors, then if any factor be common to all, it would be a common measure.

If the given numbers have not a *common factor* they cannot have a common measure greater than unity, and consequently are either prime numbers or are prime to each other. (Arts. 7 and 12.)

EXAMPLE.—Find a common divisor of 14, 35, and 63.

14=2×7; 35=5×7, and 63=3×3×7. The factor 7 is common to all the given numbers, and is therefore a common measure of them.

EXERCISE 27.

1. Find a common divisor of 21, 18, 27 and 36. *Ans.* 3.
2. Find a common divisor of 21, 77, 42 and 35. *Ans.* 7.
3. Find a common divisor of 26, 52, 91 and 143. *Ans.* 13.
4. Find a common divisor of 82, 118 and 146. *Ans.* 2.

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26. To find the greatest common measure of two quantities—

RULE.

Divide the larger by the smaller; then the divisor by the remainder; next the preceding divisor by the new remainder:—continue this process until nothing remains, and the last divisor will be the greatest common measure. If this be unity, the given numbers are prime to each other.

EXAMPLE.—Find the greatest common measure of 3252 and 4248.

```
3252)4248(1
     3252
     ----
      996)3252(3
          2988
          ----
           264)996(3
               792
               ---
               204)264(1
                   204
                   ---
                    60)204(3
                       180
                       ---
                        24)60(2
                           48
                           --
                           12)24(2
                              24
```

996, the first remainder, becomes the second divisor; 264, the second remainder, becomes the third divisor, &c. 12, the last divisor, is the required greatest common measure.

PROOF.—In order to establish the truth of this rule, it is necessary to remember (19-XIII. and XIV.) that if one number measure another it will likewise measure any integral multiple of that other; and if one number measure two others, it will also measure their sum or their difference.

First, then, 12 is a common measure of 8252 and 4248. Beginning at the end of the process; because 12 measures 12, it also measures 24, a multiple of 12; because 12 measures 24, it measures 48, a multiple of 24; because 12 measures 12 and also 48, it measures 60, which is their sum; because 12 measures 60, it measures 180, a multiple of 60; because 12 measures 180, and also 24, it measures their sum, which is 204; because 12 measures 204, and likewise 60, it measures their sum, 264; because 12 measures 264, it measures 792, a multiple of 264; and because 12 measures 792, and also 204, it measures their sum, which is 996; because 12 measures 996, it measures 2988, a multiple of 996; and because 12 measures 2988, and also 264, it measures their sum, 3252; and because 12 measures 3252, and also 996, it measures their sum, which is 4248. 12, therefore, measures each of the given numbers, and is a common measure; next it is their *greatest* common measure.

For, if not, let some other as 13, be greater. Then, (beginning now at the top of the process) because 13 measures 3252, and also 4248, it measures their difference, which is 996; because 13 measures 996, it measures 2988, a multiple of 996, and because 13 measures 3252, and also 2988, it also measures their difference, which is 264; because 13 measures 264, it also measures 792 a multiple of 264; and because 13 measures 792, and also 996, it measures their difference, which is 204; because 13 measures 264, and also 204, it measures their difference, which is 60; because 13 measures 60, it measures 180, a multiple of 60; and because 13 measures 180, and also 204, it measures their difference, which is 24; because 13 measures 24, it measures 48, a multiple of 24; and because 13 measures 60, and also 48, it measures their difference, which is 12. That is, 13 measures or divides 12—a greater number measures a less, which is impossible.

Therefore 13 is not a common measure of 3252 and 4248; and in a similar manner it may be shown that no number greater than 12 is a common measure. Therefore 12 is the greatest common measure.

As the rule might be proved for any other example equally well, it is true in all cases.

EXERCISE 28.

1. What is the greatest common measure of 296 and 407? *Ans.* 37.
2. What is the greatest common measure of 506 and 308? *Ans.* 22.
3. What is the greatest common measure of 74 and 84? *Ans.* 2.
4. What is the greatest common measure of 1825 and 2555? *Ans.* 365.
5. What is the greatest common measure of 556 and 672? *Ans.* 4.

---

27. To find the greatest common measure of more than two numbers—

RULE.

Find the greatest common measure of two of them; then, of this common measure and a third; next of this last common measure and a fourth, &c. The last common measure found will be the greatest common measure of all the given numbers.

EXAMPLE 1.—Find the greatest common measure of 679, 5901, and 6734.

By the last rule we find that 7 is the greatest common measure of 679 and 5901; and by the same rule that it is the greatest common measure of 7 and 6784 (the remaining number), for 6784÷7=969, with no remainder. Therefore 7 is the required number.

EXAMPLE 2.—Find the greatest common measure of 936, 736, and 142.

The greatest common measure of 936 and 736 is 8, and the greatest common measure of 8 and 142 is 2; therefore 2 is the greatest common measure of the given numbers.

This rule may be shown to be correct in the same way as the last; except that in proving the number found to be a *common* measure, we are to begin at the end *of all* the processes, and go through all of them in succession; and in proving that it is the *greatest* common measure, we are to begin at the commencement of the first process, or that used to find the common measure of the two first numbers, and proceed successively through *all*.

EXERCISE 49.

1. What is the greatest common measure of 110, 140, and 680? *Ans.* 10.
2. What is the greatest common measure of 1326, 3094, and 4420? *Ans.* 442.
3. What is the greatest common measure of 468, 922, and 375? *Ans.* They have none.
4. What is the greatest common measure of 204, 1190, 1445, and 2006? *Ans.* 17.

SECOND METHOD.

28. It is manifest that the greatest common measure or greatest common divisor of two or more numbers, must be their greatest common factor, and that this greatest common factor must be the product of all the prime factors that are common to all the given numbers.

Hence to find the greatest common measure of two or more numbers, we have the following—

RULE.

Resolve each of the given numbers into its prime factors; and the product of those factors, which are common to all, will be the greatest common measure.

EXAMPLE 1.—What is the greatest common measure of 1365 and 1995?

3)1365
5)455
7)91
13

Hence, 3, 5, 7, and 13 are the prime factors.

3)1995
5)665
7)133
19

Hence, 3, 5, 7, and 19 are the prime factors.

And the factors that are common to both are 3, 5, 7. Hence 3×5×7=105 =greatest common measure.

EXAMPLE 2.—What is the greatest common measure of 108, 126, and 162?

$108=2^2\times3^3$, $126=2\times3^2\times7$, and $162=2\times3^4$.

Hence, the factors that are common are 2 and 3^2, and the greatest common measure$=2\times3^2=18$.

EXERCISE 30.

1. *Work by this method all the preceding examples.*
2. What is the greatest common measure of 56, 84, 140, 168? *Ans.* 28.
3. What is the greatest common measure of 241920, 380160, 69120, 103680? *Ans.* 34560.
4. What is the greatest common measure of 10800, 28040, and 2160? *Ans.* 40.

LEAST COMMON MULTIPLE.

29. One number is a common multiple of two or more others when it can be divided by each of them without a remainder.

30. One number is the least common multiple (l. c. m.) of two or more others when it is the *least* number that can be divided by each of them without a remainder.

31. It is evident that a dividend will contain a divisor an exact number of times, when it contains, as factors, *every* factor of that divisor; and hence, the question of finding the *least* common multiple of several numbers is reduced to finding a number which shall contain all the prime factors of each number and *none others.* If the numbers have no common prime factor, their product will be their least common multiple.

Suppose we wish to see what is the least common multiple of 9, 12, 16, 20, and 35. Resolving these into their prime factors, we obtain $9=3^2$, $12=2^2\times3$, $16=2^4$, $20=2^2\times5$, and $35=7\times5$. Now it is plain that 2^4 must enter into the least common multiple as a factor, and, since 2^4 is a multiple of 2^2, we do not consider 2^2 also a factor of the least common multiple. So also 3^2 must be a factor of the least common multiple; and since it contains 3, we do not again multiply by 3. Lastly, 5 and 7 must enter into the least common multiple.

The factors of the least common multiple are then 2^4, 3^2, 5 and 7; and these, multiplied together, give $2^4\times3^2\times5\times7=5040$=least common multiple.

Hence, to find the least common multiple of two or more numbers, we have the following—

RULE.

Resolve the numbers into their prime factors (Art. 21), *select all the different factors which occur, observing when the same factor has* *different powers, to take the highest power. The continued product of the factors thus selected will be the least common multiple.*

Exercise 31.

1. What is the least common multiple of 8, 9, 10, 12, 25, 32, 75, and 80?

Here $8=2^3$, $9=3^2$, $10=2\times5$, $12=2^2\times3$, $25=5^2$, $32=2^5$, $75=5^2\times3$, $80=2^4\times5$. Therefore the least common multiple $=2^5\times3^2\times5^2=70200$.

2. What is the least common multiple of 6, 7, 42, 9, 10, and 630?
Ans. $2\times3^2\times5\times7=630$.

3. What is the least common multiple of the nine digits?
Ans. $2^3\times3^2\times5\times7=2520$.

4. What is the least common multiple of 6, 9, 12, 15, 18, 21, and 30?
Ans. 1260.

5. What is the least common multiple of 670, 100, 335, and 25?
Ans. 6700.

6. What is the least common multiple of 8, 10, 18, 27, 36, 44, and 396?
Ans. 11880.

SECOND METHOD.

32. We may also find the least common multiple of two or more numbers by the following—

RULE.

Write the given numbers in a line, with two points between them. Divide by the LEAST *number which will divide any two or more of them without a remainder, and set the quotients and the undivided numbers in a line below.*

Divide this line and set down the results as before; thus continue the operation till there are no two numbers which can be divided by any number greater than 1.

The continued product of the divisors and the numbers in the last line will be the least common multiple sought.

Example 1.—What is the least common multiple of 16, 48, and 108?

```
2)16 . . 48 . . 108
  ─────────────────
2)8 . . 24 . . 54
  ─────────────────
2)4 . . 12 . . 27
  ─────────────────
2)2 . . 6 . . 27
  ─────────────────
3)1 . . 3 . . 27
  ─────────────────
  1 . . 1 . . 9
```

Ans. $2\times2\times2\times2\times3\times9=432$ = least common multiple.

The least common multiple of 1, 1, and 9 is 9, and the least common multiple of 1, 1, and 9 × 3, will be the least common multiple of 1, 3, and 27, the numbers of the fifth line; the least common multiple of 1, 3, and 27, ×2, will be the least common multiple of 2, 6, and 27, the numbers of the fourth line; the least common multiple of 2, 6, and 27, ×2, will be the least common multiple of 4, 12, and 27, the numbers in the third line; the least common multiple of 4, 12, and 27, ×2, will be the least common multiple of 8, 24, and 54, the numbers in the second line; and the least common multiple of 8, 24, and 54, ×2, will be the least common multiple of 16, 48, and 108, the given numbers.

The *reason* of the preceding rule depends upon the principle that the least common multiple of two or more numbers, is composed of all the *prime factors* of the given numbers, each taken the greatest *number* of times it is found in either of the given numbers.

NOTE.—In finding the least common multiple by this method, it is necessary to divide by the *smallest* number, which will divide two or more of them without a remainder, because the divisor may otherwise be a composite number (Art. 21), and have a factor *common* to it, and one of the quotients in the last line. Consequently the continued product of the divisors and these quotients or undivided numbers in the last line, would be too great for the least common multiple.

Thus in the third of the following operations the divisor 9 is a composite number, containing the factor 3, common to it and the 3 in the quotient: consequently the product is *three times too large*. In the second operation the divisor 12 is a composite number, and contains the factor 6 common to it, and the 6 in the quotient: therefore the product is *six times too large*.

The object of arranging the given numbers in a line, is that all of them may be resolved into their prime factors at the same time; and also to present at a glance the factors that compose the least common multiple required.

EXAMPLE 2.—What is the least common multiple of 12, 18, 36?

```
       I.                    II.                    III.
2)12..18..36         12)12..18..36          2)12..18..36
  ----------             ----------            ----------
2)6.. 9..18           3)1..18.. 3           2)6.. 9..18
  ----------             ----------            ----------
3)3.. 9.. 9              1.. 6.. 1          9)3.. 9.. 9
  ----------           12×3×6=216              ----------
3)1.. 3.. 3                                     3.. 1.. 1
  ----------                                 2×2×9×3=108.
   1.. 1.. 1
2×2×3×3=36=l. c. m.
```

EXERCISE 32.

1. Find the least common multiple of 12, 20, and 24. *Ans.* 120.
2. Find the least common multiple of 14, 21, 3, 2, and 63. *Ans.* 126.
3. Find the least common multiple of 18, 12, 39, 216, and 234. *Ans.* 2808.
4. Find the least common multiple of 8, 18, 15, 20, and 70. *Ans.* 2520.
5. Find the least common multiple of 24, 16, 18, and 20. *Ans.* 720.
6. Find the least common multiple of 60, 50, 144, 35, and 18. *Ans.* 25200.
7. Find the least common multiple of 27, 54, 81, 14, and 63. *Ans.* 1134.

THIRD METHOD.

33. The least common multiple of several numbers is most expeditiously found by the following—

RULE.

Write the given numbers in a line; take any one of them as divisor, and strike out of each of the given numbers all the factors that are common to it and the assumed number.

Arrange the uncancelled factors of the given numbers, and the uncancelled numbers in a line; take the least other number which exactly contains one or more of them, and strike out all the factors of the numbers in the second line which are common to any of them and the second assumed number.

Proceed thus until the assumed numbers cancel all the factors of the given numbers.

Multiply all the assumed numbers together for the least common multiple of the given numbers.

Example 1.—What is the least common multiple of 16, 27, 45, 60, 88, 96, 100?

$$\begin{array}{r|l}
\text{Assume } 100 & \cancel{16}..27..\cancel{45}..\cancel{60}..88..\cancel{96}..\cancel{100} \\
\text{Assume } 24 & 4..\cancel{27}..\;9..\;\cancel{3}..\cancel{22}..\cancel{24} \\
\text{Assume } 99 & \qquad\cancel{9}..\;\cancel{3}..\qquad\cancel{11} \\
 & 100\times24\times99=237600=\text{l. c. m.}
\end{array}$$

Explanation.—4, a factor of 100, reduces 16 to 4, 88 to 22, and 96 to 24; 5, another factor of 100, reduces 45 to 9; and 20, another factor of 100, reduces 60 to 3. The numbers in the second line then are 4, 27, 9, 3, 22, and 24. We assume 24, of which a factor, 4, cancels 4; another factor, 2, reduces 22 to 11; and another factor, 3, reduces 27 to 9 and 9 to 3. The numbers in the third line then are 9, 3, and 11. For this line we assumed 99, of which a factor, 3, cancels 3; another factor, 9, cancels 9; and a third, 11, cancels 11.

Now since the least common multiple of a series of numbers is a number which still contains all the prime factors of each number, and none others, it is manifest that the least common multiple of the given numbers will be the same as the least common multiple of 100, and 4, 27, 9, 3, 22, and 24, because only those factors which were common to the given numbers and 100 were struck out.

Similarly, the least common multiple of 100, 24, and 9, 3, and 11, will be the same as the least common multiple of 100, and the numbers in the second line, since only those factors which were common to 24 and the numbers of the second line are struck out.

Finally the least common multiple of 100, 24, and 99, is equal to the least common multiple of the given numbers.

Example 2.—What is the least common multiple of 120, 40, 39, 65, 88, and 16?

$$\begin{array}{r|l}
\text{Assume } 120 & \cancel{120}..\cancel{40}..\cancel{39}..\cancel{65}..88..16 \\
\text{Assume } 13 & \qquad\qquad\;\;\cancel{13}..\cancel{13}..11..\;2 \\
\text{Assume } 22 & \qquad\qquad\qquad\quad\;\cancel{11}..\quad\;\cancel{2} \\
 & 120\times13\times22=34320=\text{l. c. m.}
\end{array}$$

Explanation.—We first assume 120. Now this cancels 120 and 40. Also 3, a factor of 120, reduces 39 to 13, and 5, another factor, reduces 65 to 13. Also 8, another factor, reduces 88 to 11 and 16 to 2. Next assume 13; this cancels 13 and 13. Next assume 22, of which 11, one factor, cancels the 11, and another factor, 2, cancels 2.

EXAMPLE 3.—Find the least common multiple of 12, 16, 20, 24, 30, 48, 56, and 64.

$$\begin{array}{r|l} \text{Assume } 96 & \cancel{12}\,..\,\cancel{16}\,..\,\cancel{20}\,..\,\cancel{24}\,..\,\cancel{30}\,..\,\cancel{48}\,..\,\cancel{56}\,..\,\cancel{64} \\ \text{Assume } 70 & \qquad\quad\ \ \cancel{5} \qquad\qquad \cancel{5}\,..\quad ..\,\cancel{7}\,..\,\cancel{2} \\ & 96 \times 70 = 6720 = \text{l. c. m.} \end{array}$$

EXERCISE 33.

1. What is the least common multiple of 300, 200, 150, 50, 60, 75, and 125? *Ans.* 3000.

2. What is the least common multiple of 20, 60, 15, 165, 210, 63, and 27? *Ans.* 41580.

3. What is the least common multiple of 12, 132, 144, 60, 96, and 1728? *Ans.* 95040.

Work also by this method all the preceding questions in least common multiple.

DIFFERENT SCALES OF NOTATION.

34. The *radix* or *base* of a scale of notation is its common ratio. Thus in our system the *radix* is 10; in the duodecimal system the radix is 12, &c.

35. If the expression 12345 represents a number in the common or decimal scale of notation, we read it twelve thousand three hundred and forty-five; but if it expresses a number in any other scale, we cannot so read it, because the names *thousands*, *hundreds*, &c., belong only to the decimal scale. In order to read it properly in any other scale, we should have to *invent* names for the different orders. In place, however, of doing this, we simply read over the digits and indicate the scale. For example, if the expression 24678 be a number in the *nonary* scale, we read it thus—*two, four, six, seven, eight* in the *nonary* scale.

36. We may express the number 4578 (decimal scale) by writing the order of each digit beneath it, thus,

$$\begin{array}{cccc} 4 & 5 & 7 & 8 \\ 10^3 & 10^2 & 10 & \end{array}$$

and then read it 8 units, 7 of the order of tens, 5 of the order of hundreds or tens squared, or second order of tens, 4 of the third order of tens, &c. Similarly if 4578 express a number in the *nonary* scale, we may write it,

$$\begin{array}{cccc} 4 & 5 & 7 & 8 \\ 9^3 & 9^2 & 9 & \end{array}$$

and read it 8 units, 7 *nines*, 5 of the *second order* of *nines*, 4 of the *third order* of nines, &c.

37. The expression 10 always represents the radix of the scale. In the *decimal* scale 10 is equal ten; in the *binary* scale 10 is equal two; in the *undenary* scale 10 is equal eleven, &c.

38. It is obvious that, in any scale, the highest digit used must be *one less* than the radix. Thus, in the *decimal* scale, the highest digit is 9; in the *ternary*, 2; in the *octenary*, 7, &c. In writing numbers in the duodenary scale we use the letter *t* to represent *ten*, and *e*, *eleven*; and in the undenary scale *t* likewise represents *ten*.

39. Let it be required to reduce 337 from the *decimal* to the *octenary* scale.

```
OPERATION.
 8)337
 ————
 8)42—1
  —
   5—2
```

EXPLANATION.—If we divide 337 by 8, we distribute it into 42 groups of 8 each, and have a remainder of 1 unit. If now we divide these groups of 8 by 8, we obtain 5 groups of a still higher order, each containing 8 of the former groups, with a remainder of 2 of these groups.

337 in the decimal scale, is therefore equal to 521 in the octenary scale; i. e., the successive remainders written in order constitute the equivalent expression in the required scale.

Hence, to reduce a number from one scale to another, we have the following—

RULE.

Divide the number continually by the radix of the proposed scale, till the quotient is less than the radix.

Write all the remainders, thus obtained, in regular order from left to right, beginning with the last, and placing 0s where there are no remainders. The result will be the required number.

EXAMPLE 1.—Reduce 7342 from the common to the *quinary* scale.

```
OPERATION.
 5)7342
  ————
 5)1468—2
  ————
 5)293—3
  ———
 5)58—3
   —
 5)11—3
  —
   2—1
```

Therefore 7342 *denary*=213332 *quinary*.

EXAMPLE 2.—Express nine millions, three hundred and forty-two thousand and twenty-seven, in the *duodenary* scale.

OPERATION.
12)9342027
12)778502—3
12)64875—2
12)5406—3
12)450—6
12)37—6
3—1

Therefore 9342027 *denary*=3166323 *duodenary.*

EXERCISE 34.

1. Change 592835 from the *decimal* to the *duodenary* scale. *Ans.* 2470*te*.
2. Express the common number 3700 in the *quinary* scale. *Ans.* 104300.
3. Express 10000 in the *undenary* scale. *Ans.* 7571.
4. Express a million in the *senary* scale. *Ans.* 33233344.
5. Express 10000 in the *octenary* scale. *Ans.* 23420.
6. Transform 12345654321 into the *duodenary* scale. *Ans.* 248664*et*69.
7. Express 10000 in the *nonary* scale. *Ans.* 14641.
8. Transform 300 from the common to the *binary* scale. *Ans.* 100101100.

EXAMPLE 1.—Transform 2313042 from the *quinary* to the *octenary* scale.

OPERATION.
V.
8)2313042
8)131310—7
8)10100—5
8)311—2
8)20—1
1—2

EXPLANATION.—We divide here as before, bearing in mind, however, that the ratio is no longer ten, but *five*. We proceed thus.—8 in 2, no times; twice five (the radix) is ten, and 3 make thirteen; 8 in 13, 1 and 5 over; 5 times 5 are 25, and 1 make 26; 8 in 26, 3 times and 2 over; twice 5 are 10, and 3 make 13, 8 in 13, once and 5 over, &c.

Therefore 2313042 *quinary*=121257 *octenary.*

NOTE.—The Roman Numeral written over the number indicates the radix of the scale.

Example 2.—Transform 378*t*13 from the *undenary* to the *duodenary* scale.

OPERATION.
XI.
12)378*t*13
12)34456—8
12)3132—4
12)294—9
12)26—9
12)2—4

Observe the first two figures here are not thirty-seven, but 3×11+7=40. We say 12 into 40, 3 times and 4 over; next, 12 into 4×11+8=or 52, &c.

378*t*13 *undenary*=249948, *duodenary*. *Ans.*

Example 3.—Transform *t*423*t* from the *duodenary* to the *nonary* scale.

OPERATION.
XII.
9)*t*423*t*
9)11971—1
9)1649—4
9)206—3
9)28—6
3—5

Observe, here we say 9 into *t* ten, 1 and 1 over; 9 into 16, (1×12+4) 1 and 7 over; 9 into 86, (7×12+2) 9 and 5 over; 9 into 63, (5×12+3) 7; 9 into *t*, 1 and 1 over.

And we proceed in the other lines in the same manner.

*t*423*t* *duodenary*=356341 *nonary*.

Exercise 35.

1. Transform 37704 from the *nonary* to the *octenary* scale. *Ans.* 61415.
2. Transform 444 and 4321 from the *quinary* to the *septenary* scale. *Ans.* 235 and 1465.
3. Transform 1212201 from the *quaternary* to the *nonary* scale. *Ans.* 10000.

---

40. A number may be transformed from any scale to the decimal by the preceding rule, but the following is more convenient.

Multiply the left hand figure by the given radix, and to the product add the next figure.

Then multiply this sum by the radix and add the next figure. Continue this process until all the figures have been used. Then the last product will be the number in the decimal scale.

Note.—Both this and the preceding rule are the same in principle as reducing denominate numbers from one denomination to another.

Example 1.—Reduce 76345 from the *octenary* scale to the *decim*
scale.

Operation.
VIII.
76345
8
—
62 of the fourth order.
8
—
499 of the third order.
8
—
3996 of the second order.
8
—
31973 units=required number in *decimal* scale.

Example 2.—Transform *ettete* from the *duodenary* to the common or *decimal* scale.

Operation.
XII.
ettete
12
—
142=number of fifth order.
12
—
1714=number of fourth order.
12
—
20579=number of third order.
12
—
246958=number of second order.
12
—
2963507=units=required number in *decimal* scale.

Exercise 36.

1. Change 20212331 from the *quaternary* into the *decimal* scale. *Ans.* 35261.
2. Change 101202220 from the *ternary* into the *decimal* scale. *Ans.* 7854.
3. Transform 1522365 from the *nonary* into the *decimal* scale. *Ans.* 841568.
4. Transform 33233344 from the *senary* into the *decimal* scale. *Ans.* 1000000.

Example 5.—Transform 2734, *octenary* scale, into the *undenary*, *septenary*, and *quinary* scales, and prove the results by reducing a four numbers to the *decimal* scale.

VIII.	VIII.	VIII.
11)2784	7)2784	5)2784
11)210—4	7)326—2	5)454—0
11)14—4	7)36—4	5)74—0
1	4—2	5)14—0
		2—2

herefore 2784 *octenary*=1144 *undenary*=4242 *septenary*=22000 *quin.* *y.*

8	11	7	5
23	12	30	12
8	11	7	5
187	135	214	60
8	11	7	25
1500 *denary.*	1500 *denary.*	1500 *denary.*	1500 *denary.*

Since the results all agree when reduced to the denary scale, we conclude e work is correct.

6. Transform 132713 *nonary*, into the *ternary*, *duodenary*, and *tenary* scales, and prove the results by reducing all four numbers to e *denary* scale.

7. Transform *t*2*t*290 *duodenary*, into the *nonary*, *senary*, *quater-ry*, and *binary* scales, and prove the result by reducing all five mbers to the *decimal* scale.

FUNDAMENTAL RULES.

41. The fundamental rules of arithmetic are carried on the different scales as with numbers in the ordinary or demal scale; observing that, when we wish to find what to *rry* in addition, subtraction, multiplication, &c., we divide, t by *ten*, but by the radix of the particular scale used.

Example 1.—Add together 34120, 3121, 13102, 31410, 12314, 2243 and 444444 in the *senary* scale.

ration.
VI.
34120
3121
13102
31410
12314
12243
44444
44042 *Ans.*

Observe the sum of the first line is 14, which, divided by 6, the radix of the scale, gives us 2 to set down and 2 to carry; the sum of the second line is 16, which, divided by the radix, 6, gives us 4 to set down and 2 to carry, &c.

Example 2.—From 48*t*76 take 9*t*09, in the *undenary* scale.

ration.
XI.
48*t*76
9*t*09
068

Observe, here we say 9 from 6, we cannot, but 9 from 17 (1 borrowed=11 and 6) and 8 remains, &c.

EXAMPLE 3.—Multiply 3426 by 567, in the *octenary* scale.

```
OPERATION.
  VIII.
  3426
   567
 -----
 30632
25204
21556
-------
2460472 Ans.
```

Observe, we say 7 times 6 are 42, 8 (the radix) into 42 5 to carry and 2 to set down; 7 times 2 are 14 and 5 make 19, equal to 3 to set down and 2 to carry, &c.

EXAMPLE 4.—Divide 671884 by 7876, in the *nonary* scale.

```
OPERATION.
 IX.  IX.
7876)671884(75 7501/7876 Ans.
     61786
     -----
      52424
      43823
      -----
       7501
```

Here 7876 will go into 67188 7 times (observe it would go 8 times in the *decimal* scale); and 7876 multiplied by 7 gives 61786, this being subtracted, gives a remainder, 5242, to which we bring down the next digit, 4, and proceed as in common division.

NOTE.—After the units' figure is brought down, we may either write the remainder in the form of a fraction, as in example 29, or we may place a point, and annexing 0s, continue the division as in the following example.

Observe, this point is called the *decimal* or *denary point* only in the decimal system. In every other scale of notation it takes its name from the system—thus, in the duodenary or duodecimal system it is called the *duodenary* or *duodecimal* point, in the senary system, the *senary* point, &c.

EXAMPLE 5.—Divide t134567 by e478, in the *duodenary* scale.

```
      OPERATION.
  XII.    XII.
e478)t134567(t7t·1e, &c.
     95t06
     -----
      758e6
      67829
      -----
       97897
       95t06
       -----
        1t91·0
        e47·8
        ------
        e45·90
        t52·79
```

EXERCISE 37.

1. Multiply 252 by 252, in the *senary* scale. *Ans.* 122024.
2. Divide 32e75721 by 62tt, in the *duodenary* scale. *Ans.* 62te.
3. From 201210 take 102221, in the *ternary* scale. *Ans.* 21212.
4. Multiply 57264 by 675, in the *octenary* scale. *Ans.* 51117344.
5. Add together 101, 1001, 1111, 1011, 1000, 1111, and 10101, in the *binary* scale. *Ans.* 1010100.

6. Divide 142613 by 2143, in the *septenary* scale. *Ans.* 50·5254+.
7. Add together 65432, 43210, 1444, 65001, and 54321, in the *septenary* scale. *Ans.* 326041.
8. From 7*t*348 take 5*e*6*t*4, in the *duodenary* scale. *Ans.* 1*t*864.
9. Multiply 84*t*7 by 6666, in the *duodenary* scale. *Ans.* 1*t*36*e*296.
10. Divide 1010100001 by 100101, in the *binary* scale. *Ans.* 10010 $\frac{111}{100101}$.

42. All the methods of proof given in Sec. II., for the fundamental rules in the common scale, apply to the various other scales; but it must be remembered that, in using the principle of the proof by *nines* for multiplication and division, we use, not *nine*, but a number one less than the radix of the scale.

Thus, in applying this principle to the proof in Example 4, *sevens* cast out of 57264, give a remainder 8; sevens cast out of 675, give a remainder 4, 4×8, and *sevens* cast out, give a remainder 5; sevens cast out of 51117844, give a remainder 5.

If the radix be 12, we cast out the 11s; if the radix be 6, we cast out the 5s, &c.

43. Numbers containing digits to the right of the separating point, are dealt with according to the rules given in Arts. 53 and 88, Sec. II.

Example.—Multiply 87·14*t*8 by 6·1*et*, in the *duodenary* scale.

OPERATION.
XII.
87·14*t*8
6·1*et*

2*ee*2066
8868549
8714*t*8
1968516

1*t*1·*t*08*e*886

We place the separating point in the product so as to have seven digits to the right of it, because there are four to the right of the point in the multiplicand and three in the multiplier, and 4+3=7. (Art. 53, Sect. II.)

DUODECIMAL MULTIPLICATION.

44. The term *duodecimal* is commonly applied to a set of denominate fractions having 1 foot (*linear, square,* or *cubic* measure) for their unit.

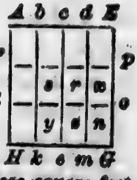
The foot is supposed to be divided into 12 equal parts, called *primes*; each of which is divided into 12 equal parts, called *seconds*, &c.

TABLE.

12 fourths''''	make	1 third,	marked	'''
12 thirds	"	1 second,	"	''
12 seconds	"	1 prime,	"	'
12 primes	"	1 foot,	"	ft.

45. The term "inch," sometimes used in this table, is objectionable, corresponding to "prime" only when the unit is a linear foot. When the unit is a square foot, the prime is $\frac{1}{12}$ of a square foot, or is a *surface* 12 inches long and 1 inch wide; when the unit is a cubic foot, the prime is $\frac{1}{12}$ of a cubic foot, or is a *solid* 12 inches long, 12 inches wide, and 1 inch thick.

46. Let *AEHG* represent the surface of a rectangular table *four* feet in length and *three* in breadth. Now, if *AE* be divided into four equal parts, and *AH* into three equal parts, each of these parts, *Ab*, *bc*, *fl*, &c., will be 1 foot long, and if lines *bk*, *ce*, *dm* are drawn through *b*, *c*, and *d*, parallel to *AH*, and lines *fp*, *lo* through *f* and *l*, parallel to *AE*, they will divide the whole surface into the small figures, *Absf*, *bsrc*, &c.



And, since *Ab*=1 foot, and *Af*=1 foot, *Afsb* is a *square foot*, so likewise is each of the other figures, *bsrc*, *crwd*, &c.

Now it is evident that there are as many vertical rows of these square feet as there are linear feet in *AE*, and as many squares in each row as there are linear feet in *AH*, that is in this case the number of *square feet* in the surface =4 × 3=12.

As the same method of proof would apply in any similar case, it appears that—

The area of a rectangular surface is found in square feet, and fractions of a square foot, by multiplying the number expressing how many linear feet, &c., there are in the length, by the number expressing how many linear feet, &c., there are in the breadth.

NOTE.—In linear measure, primes are linear inches; in square measure, seconds are square inches; and in cubic measure, thirds are cubic inches.

47. The example under Section 43, page 143, is, in effect, equivalent to finding the area of a rectangle, one side of which is 43 feet 1′ 4″ 10‴ and 3⁗ long, and the other 6 ft. 1′ 11″ 10‴ long. The answer may be translated 265 sq. ft. 10′ 0″ 8‴ 11⁗ 8′′′′′ 3′′′′′′ and 6′′′′′′′.

NOTE.—1t1, the number to the left of the separating point, is a number in the duodenary scale. In order to read it in common terms, we convert it to an equivalent number in the decimal scale (Art. 40), and thus obtain 265. It is obvious that, since the orders primes, seconds, thirds, &c., form a series of numbers descending in a 12-fold proportion from left to right, we must allow the digits to the right of the point to remain as they are.

EXAMPLE.—Find the area of a rectangular ceiling 43 ft. 4′ 7″ long by 20 ft. 11′ 10″ wide.

Here, since 43 and 20 are numbers in the common scale, we must reduce them to the duodenary scale before attaching them by the point to the other parts of the numbers. We thus obtain for the first, 37, and for the second, 18. After multiplying and pointing off four places in the product, we find 63t to the right of the point; this, reduced to an equivalent number in the common scale, gives us 910, to which we attach the other four digits, with their indices, as below.

```
OPERATION.
   XII.
  37·47
  18·et
 ------
  3019t
 33925
24e08
3747
-------
63t.502t=910 sq. ft. 5′ 0″ 2‴ 10⁗   Ans.
```

48. The common arithmetical rule for duodecimal multiplication is as follows:—

RULE.

Write the multiplier under the multiplicand having quantities of the same denomination under each other.

Multiply each term of the multiplicand by each term of the multiplier separately.

Write the partial products under one another, so as to have quantities of the same name in the same vertical column, and add the several partial products together.

NOTE.—Considering the foot to have no index, the denomination of the product of any two factors is found by adding their indices.

Thus, 3'' × 2''' give 6'''''; 4 ft. × 7'''' give 28''''; 2 ft. × 3 ft. give 6 ft.; 9' × 11 give 99'', &c.

This is commonly expressed, for the sake of brevity, by saying—feet into feet produce feet, feet into primes produce primes, &c., primes into feet produce primes, primes into primes produce seconds, &c., seconds into seconds produce fourths, seconds into thirds produce fifths, &c.

EXAMPLE 1.—Multiply 43 ft. 4' 7'' by 20 ft. 11' 10''.

OPERATION.

43	4'	7''		
20	11	10		
3	0	1	9'''	10''''
39	9	2	5	
867	7	8		
910	5'	0''	2'''	10''''

Here 7 and 10 multiplied together, give us 70, and adding their indices, we see that the product is so many fourths—70'''', are equal to 10'''' to set down and 5''' to carry. Next 4' × 10'' = 40''' and 5''' make 45''' = 3'' 9''', &c.

49. In comparing this example with the previous number it will be seen that the two methods very closely agree—the only difference being that, in the latter method, upon reaching the units or feet, we drop the duodecimal scale and carry on the process in the decimal scale, while, in the former, we carry on the whole process in the duodecimal scale, and afterwards reduce that part of the expression to the left of the separating point to the common or decimal scale.

50. Provided we multiply every part of the multiplicand by every part of the multiplier, it is perfectly immaterial where we commence the process. It is customary, however, to commence, not as we have done in the last example, with the lowest denomination of both multiplier and multiplicand, but with the highest of the multiplier and the lowest of the multiplicand. Hence duodecimal multiplication is frequently called Cross Multiplication.

Example 2.—Multiply 8 ft. 2′ 7″ 4‴ by 1′ 3″ 7‴

OPERATION.

8 ft.	2′	7″	4‴			
	1	3	7			
8	2	7	4‴			
	9	7	10	0⁗		
	1	10	6	8	4′′′′′	
4′	2″	1‴	8⁗	8′′′′′	4′′′′′′	*Ans.*

Exercise 38.

1. Multiply 4 ft. 7′ 6″ 10‴ by 9 ft. 7′ 11″ 11‴.
   Ans. 44 sq. ft. 9′ 1″ 8‴ 0⁗ 5′′′′′ 2′′′′′′.
2. Multiply 19 ft. 10′ 3″ by 11 ft. 2′ 7″. *Ans.* 222 sq. ft. 8′ 0″ 5‴ 9⁗.
3. Multiply 9″ 7‴ 4⁗ by 7‴ 3⁗ 11′′′′′.
   Ans. 5⁗ 10′′′′′ 4′′′′′′ 11′′′′′′′ 8′′′′′′′′ 8′′′′′′′′′.
4. How many square inches, &c., are there in a sheet of paper 9¾ inches and 5 inches 7″ 4‴ wide? *Ans.** 4′ 6″ 8‴ 6⁗ or $54\frac{17}{24}$ sq. inches.
5. What is the superficial contents of a sheet of glass whose length is 7 ft. 4′ 11″ and breadth 3 ft 2′ 2″? *Ans.* 23 sq. ft. 6′ 9″ 7‴ 10⁗.

51. The solid contents are found by multiplying together the length, breadth, and thickness.

Example.—How many cords of wood are there in a pile 79 ft. 8 inches long, 4 ft. 2 inches wide, and 7 ft. 11 inches high?

OPERATION.

FIRST METHOD.

t7·8
4·2
1184
3268
237·e4
7·e
214348
141774
No. of ft. in cord=t8)162e·t88(18·64469 *duodenary* = 20[illegible] com. scale.
t8
76e
714
57·t
54·0
3·t3
3·68, &c.

SECOND METHOD.

79 8′
4 2′
18 3′ 4″
318 8′
331 11′ 4″
7 11′
304 3′ 4″ 8‴
2323 7′ 4″
2627 10′ 8″ 8‴÷128.
(number of ft. in cord)
=20[illegible] cords. *Ans.*

---

* $\frac{4}{12}+\frac{6}{144}+\frac{8}{1728}$, &c., *of a square foot.*

Exercise 39.

1. Multiply together 15 ft., 1 ft., 1 ft. 2′, and 8′.
Ans. 11 cubic ft. 8′=11 cubic ft. 1152 cubic in.
2. Multiply together 53 ft. 6 in., 10 ft. 3 in., and 2 ft.
Ans. 1096 cubic ft. 9′.
3. How many cords of wood in a pile 10 ft. long, 5 ft. high, and 7 ft. wide?
Ans. 2 cords 94 cubic ft.
4. How many cords of wood are there in a pile 4 ft. wide, 5 ft. 3 in. high, and 70 ft. long?
Ans. $11\frac{31}{64}$.
5. What are the exact cubic contents of a block of marble 4 ft. 7′ 8″ long by 9 ft. 6′ wide and 2 ft. 11′ thick?
Ans. 128 cubic ft. 6′ 5″ 2‴.
6. How many bricks, 8 inches long, 4 inches wide, and 2 inches thick, will it require to make a wall 25 ft. long, 20 ft. high, and 2 ft. 6 inches thick?
Ans. 33750 bricks.

---

52. It is sometimes asked how we can multiply feet, inches, &c., by feet, inches, &c., while we cannot multiply pounds, shillings and pence by pounds, shillings and pence. The answer is very simple.

1st. When we say that feet multiplied by feet give square feet, we merely use, as we have seen, (Art. 46), an abbreviated form of expression for the following, viz: that "the number of square feet contained in any rectangular surface, is equal to the product of two numbers, one of which represents the number of linear feet in one side; and the other the number of linear feet in the adjacent side."

2nd. When we are multiplying together primes, seconds, &c., we are merely multiplying together a set of factors having 12 or powers of 12 for denominators; and when we say that *seconds* multiplied by *fourths*, give *sixths*; *primes*, multiplied by *seconds*, give *thirds*, &c., we simply mean that the product of any two of these fractions is a fraction having for its denominator a power of 12, which power is indicated by the sum of the indices of the factors.

It is hence obvious that duodecimal multiplication affords no support whatever to the idea that money may be multiplied by money.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*The numbers after the questions refer to the articles of the Section.*

1. What is the *measure* of a number? (1)
2. What is the *multiple* of a number? (2)
3. What is an *integer*? (3)
4. Of how many kinds are integers? (4)
5. What is an *even* number? (5)
6. What is an *odd* number? (6)
7. What is a *prime* number? (7)
8. What is a *composite* number? (8)
9. What are the *factors* of a number? (9)
10. By what other names are factors known? (10)
11. What is a *common measure* of two or more numbers? (11)
12. When are two or more numbers *prime to each other*? (12)
13. Are all prime numbers prime to each other? (12)
14. Are all composite numbers prime to each other? (12)
15. What are *commensurable* numbers? (13)

16. What are *incommensurable* numbers? (14)
17. What is a *square* number? (15)
18. What is a *cube* number? (16)
19. What is a *perfect* number? (17)
20. Mention some perfect numbers. How do all perfect numbers terminate? (17)
21. What are *amicable* numbers? Mention some amicable numbers. (18)
22. What is meant by the *properties of numbers?* (19)
23. What is the sum of two or more even numbers? (19-I.)
24. What is the difference of two even numbers? (19-II.)
25. What is the sum of 3, 5, 7, &c., odd numbers? (19-IV.)
26. What is the sum of 2, 4, 6, 8, &c, odd numbers? (19-V.)
27. What is the sum or difference of an odd and an even number? (19-VI.)
28. When is the product of any number of factors even? (19-IX.)
29. When is the product of any number of factors odd? (19-XI.)
30. When will a number measure the *sum*, *difference* and product of two numbers? (19-XIII.)
31. If the number 9 be multiplied by any single digit to what is the sum of the digits in the product equal? (19-XVI.)
32. By what is any number ending in 0 divisible? 19-XIX, &c.)
33. By what is any number ending in 5 divisible? (19-XX.)
34. By what is any number ending in 2 divisible? (19-XIX.)
35. When is a number divisible by 4? (19-XXII.)
36. When is a number divisible by 8? (19-XXIII.)
37. When is a number divisible by 9? (19-XXIV.)
38. When is a number divisible by 3? (19-XXV.)
39. When is a number divisible by 11? (19-XXVI.)
40. Show that every composite number may be resolved into prime factors. (19-XXVII.)
41. Show that the *least* divisor of any number is a prime number. (19-XXVIII.)
42. With what digits must all prime numbers except 2 and 5 terminate? (19-XXXI.)
43. How do you find the prime numbers between any limits? (20)
44. What is this process called and why? (20)
45. When it is required to ascertain whether a given number is prime or not, what is the first thing we do? (20)
46. When we try the primes of the table as divisors, which is the highest we need use? (20)
47. Why is it unnecessary to try any divisor greater than the square root of the number? (20)
48. How do we resolve a composite number into its prime factors? (21)
49. By what numbers can a composite number be divided? (21-Note.)
50. What is the rule for finding all the divisors of a number? (22)
51. How do we find simply *how many* divisors a number has? (23)
52. What is the greatest common measure of two or more numbers? (24)
53. How do we find a common measure of two or more numbers? (25)
54. How do we find the greatest common measure of two numbers? (26)
55. Prove the rule in Art. 26.
56. How do we find the G. C. M. of three or more numbers? (27)
57. What is the second method of finding the G. C. M.? (28)
58. Upon what principle does this method rest? (28)
59. What is a common multiple of two or more numbers? (29)
60. What is the least common multiple of two or more numbers? (30)
61. Give the first rule for finding the l. c. m. of two or more numbers. (31)
62. Give the second rule. (32) What is the reason of this rule? (32)
63. Give the most convenient and expeditious rule for finding the l. c. m. of several numbers. (33)
64. What is meant by the *radix* or *base* of a system of notation? (34)
65. How do we read numbers in different scales? (35)
66. Express the number 234213 *quinary* as in Art. 36.
67. What does the expression 10 always represent? (37)

68. What is the highest digit used in any scale? (38)
69. How do we reduce a number from one scale to another? (39)
70. What is the rule for transforming a number from any scale into the decimal? (40)
71. How are the fundamental operations carried on in the different scales? (41)
72. How is the separating point named in the different scales? (41-Note.)
73. How are operations in the different scales proved? (42)
74. What are duodecimals? (44)
75. Give the table of duodecimals. (44)
76. What is a prime? (45)
77. How is the area of a rectangular surface found? (46)
78. What is the rule for duodecimal multiplication? (48)
79. How may the rule for finding the denomination of the product be concisely worded? (48)
80. How are solid contents found? (51)
81. Show that duodecimal multiplication affords no support to the idea that money may be multiplied by money, &c. (52)

EXERCISE 40.

MISCELLANEOUS EXERCISE.

On preceding rules.

1. Add together $729·18, $710·50, $166·78, £93 14s. 7⅛d., £276 19s. 10⅜d., $497·81, and £275 4s. 11¾d.
2. Multiply 47 miles 6 fur. 17 per. 4 yds. 2 ft. 7 in. by 576.
3. How many divisors has the number 243000?
4. From 713427 *octenary* take 4234434 *quinary* and give the answer in both scales.
5. Divide 79·342 by ·00006378.
6. Express 79423 and 234567 in Roman numerals.
7. What is the l. c. m. of 5, 7, 9, 11, 15, 18, 20, 21, 22, 24, 28, 30, 33, 35, 36, 40, 42, 44, 45, 48, and 50?
8. Give all the readings of 376·342.
9. Multiply 64276·3427 by 9999993000.
10. Transform 78263 *nonary* into the *quinary* and *undenary* scales and prove the results by reducing all the numbers to the *septenary* scale.
11. Form a table of all the prime numbers less than 200.
12. Reduce £672 7s. 7d. to dollars and cents.
13. What is the G. C. M. of 243000, 891, 37800 and 35100?
14. Give all the readings of 6 yards 3 qrs. 3 nails 2 inches.
15. Write down as one number, seven hundred and forty-two quintillions, nine hundred and five billions, seventy-eight thousand and fourteen, and eighty-seven million, two hundred thousand and eleven tenths of trillionths.
16. Read the following numbers—

713001002004O1·000000070402
134900101000100100·000200020002
47000000000020007·0000000000278

$30 \ 6$
12

$20 \frac{11}{12}$
12
240
11

$\frac{251}{12}$

17. Add together £178 16s. 4¾d., £97 15s. 11¼d., £698 19s. 11¾d., £216 11s. 9½d., £678 14s. 7½d., £197 13s. 11¾d., £117 6s. 5d., and £91 1s. 1¾d.
18. What are the prime factors of 276000?
19. Multiply 6 ft. 2' 7'' 9''' 10'''' by 13 ft. 11' 11'' 11''' 7''''.
20. Divide 7*te*9·047 by 718*t*96 in the *duodenary* scale.
21. What number in the common scale is the greatest that can be expressed by seven figures in the *quaternary* scale?
22. What number in the common scale is the least that can be expressed as an integral number by five figures in the *octenary* scale?
23. Reduce 74002702 square inches to acres.
24. What is the least common multiple of 240, 780, 1620, and 1728?
25. Divide $7894·16 among 3 men, 4 women, and 6 children, so that each woman shall have twice as much as a child and each man 5 times as much as a woman. What is the share of each?
26. What are the greatest and least integral numbers in the common scale that can be expressed by 10 figures in the *binary* scale?
27. Divide 729 yds. 3 qrs. 3 na. 1 in. by 7 yds. 1 qr. 1 na. 1 in.
28. Multiply 762·4978 by 63·423.
29. From 723426 take 938·9126141.
30. From 129 lb. take 63 lb. 4 oz. 7 drs. 2 scr.
31. What are the divisors of 1064?
32. How many yards of carpet 2 ft. 7 in. wide, will be required to cover a floor 30 ft. 6 in. long and 20 ft. 11 in. wide?

$\frac{61}{2} \times \frac{257}{\cancel{12}} \times \frac{4}{9} \times \frac{\cancel{12}}{31}$

SECTION IV.

VULGAR AND DECIMAL FRACTIONS, &c.

1. A fraction is an expression representing one or more of the equal parts into which any quantity may be divided.

2. If a quantity be divided into 2, 5, 9, or 34, &c., equal parts, then *one* of these parts is called *one-half*, *one-fifth*, *one-ninth*, or *one-thirty-fourth*, &c., as the case may be.

one-half is written........ $\frac{1}{2}$
one-third is written....... $\frac{1}{3}$
one-fourth is written...... $\frac{1}{4}$
one-fifth is written........ $\frac{1}{5}$
one-ninth is written....... $\frac{1}{9}$

one-hundredth is written $\frac{1}{100}$
one-sixty-eighth is written $\frac{1}{68}$
eleven-seventeenths is written $\frac{11}{17}$, &c.

3. The division of one number by another may be indicated in three different ways, viz: by using the full sign of division, ÷, or either of its parts, —, or :

Thus we may indicate the division of 17 by 8, by writing them thus 17÷8, or thus 17:8, or thus $\frac{17}{8}$.

Now the last of these, viz: $\frac{17}{8}$ is a fraction, and so in every other case, a fraction indicates the division of one number, called the *numerator*, by another number, called the *denominator*.

4. In a fraction the number below the line is called the denominator, because it indicates into how many equal parts the unit is divided,—i. e., it tells the *denomination* of the parts. The number above the line is called the numerator, because it *numerates* or tells how many of these equal parts are to be taken. (Art. 2)

5. The numerator and denominator are called the *terms* of the fraction.

25

6. Since every fraction expresses the division of the numerator by the denominator, it follows that—

The *value* of the fraction is the *quotient* obtained by dividing the numerator by the denominator.

7. Hence, 1st. When the numerator is less than the denominator, the value of the fraction is less than 1.

2nd. When the numerator is equal to the denominator, the value of the fraction is equal to 1.

3rd. When the numerator is greater than the denominator, the value of the fraction is greater than 1.

8. From (Art. 6) and (Arts. 79–84, Sect. II.) it is manifest that—

1st. Multiplying the numerator of a fraction by any number multiplies the fraction by that number.

2nd. Multiplying the denominator of a fraction by any number divides the fraction by that number.

3rd. Multiplying both numerator and denominator of a fraction by the same number does not affect the value of the fraction.

4th. Dividing the numerator of a fraction by any number divides the fraction by that number.

5th. Dividing the denominator of a fraction by any number multiplies the fraction by that number.

6th. Dividing both numerator and denominator of a fraction by the same number does not affect its value.

9. Fractions are divided into two classes:—vulgar and decimal.

10. A Decimal Fraction is a fraction in which the denominator is 1, followed by 1 or more 0s.

11. All other fractions are called Vulgar or Common Fractions.

NOTE.—The word vulgar is here used in the sense of common.

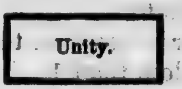
12. There are six kinds of vulgar fractions—*proper*, *improper*, *mixed*, *simple*, *compound*, and *complex*.

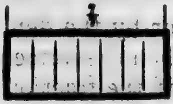
13. A Proper Fraction is one in which the denominator is greater than the numerator.

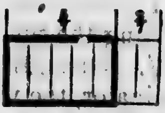
A Proper Fraction may also be defined to be a fraction whose value is less than 1.

Thus $\frac{11}{12}$, $\frac{3}{4}$, $\frac{1}{15}$, $\frac{171}{172}$, $\frac{43}{144}$, $\frac{99}{100}$ are proper fractions.

The following diagrams represent *unity*, *seven-sevenths*, and the proper fraction, five-sevenths.



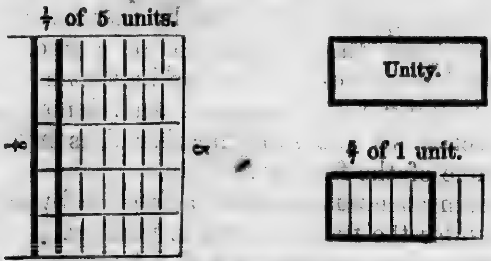




The very faint lines indicate what $\frac{5}{7}$ wants to make it *equal* to unity and *identical with* $\frac{7}{7}$. In the diagrams which are to follow, we shall, in this manner, generally subjoin the difference between the fraction and unity.

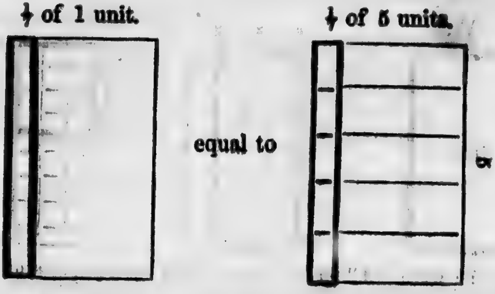
The teacher should impress on the mind of the pupil that he might have chosen any *other* unity to exemplify the nature of a fraction.

14. The following will show that $\frac{5}{7}$ may be considered as either the $\frac{5}{7}$ of 1 or the $\frac{1}{7}$ of 5, both—though not identical—being perfectly equal.



In one case we may suppose that the five parts belong to but 1 unit; in the other, that each of the five belongs to different units of the same kind.

Lastly, $\frac{1}{5}$ may be supposed as the $\frac{1}{5}$ of one unit five times as large as the former; thus—



15. An Improper Fraction is a fraction whose denominator is *not* greater than its numerator.

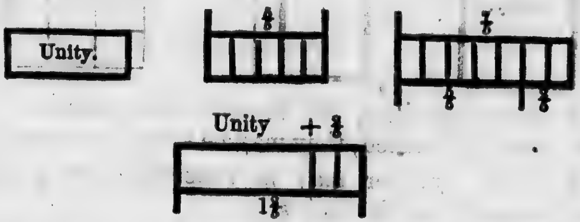
An Improper Fraction may also be defined to be a fraction whose value is equal to or greater than 1.

Thus, $\frac{3}{3}$, $\frac{12}{7}$, $\frac{5}{5}$, $\frac{11}{11}$, $\frac{222}{12}$, $\frac{141}{141}$, $\frac{5}{5}$, $\frac{18}{11}$, &c., are improper fractions.

16. A Mixed Number is a number made up of a whole number and a fraction.

Thus, 16$\frac{2}{3}$, 193$\frac{1}{7}$, 1$\frac{1}{17}$, 999$\frac{1}{5}$, 6$\frac{2}{11}$, 2$\frac{1}{4}$, &c. are mixed numbers.

17. An Improper Fraction is always equal either to a whole number or to a mixed number. The following will exemplify an improper fraction, and its equivalent mixed number:



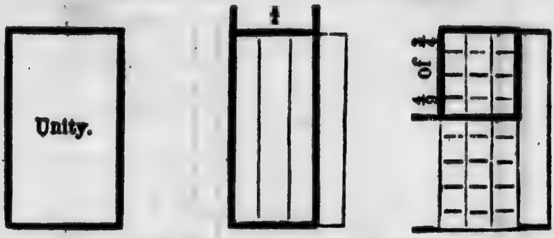
18. A Simple Fraction expresses one or more equal parts of unity.

Thus, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{11}{17}$, $\frac{5}{8}$, $\frac{199}{354}$, &c., are simple fractions.

19. A Compound Fraction expresses one or more equal parts of a fraction; or in other words, is a fraction of a fraction.

Thus, $\frac{3}{4}$ of $\frac{1}{2}$, $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{11}{13}$ of $\frac{2}{3}$ of $\frac{1}{12}$, &c., are compound fractions.

20. $\frac{4}{9}$ of $\frac{3}{4}$ means, not the four-ninths of unity, but the four-ninths of the three-fourths of unity —that is, unity being divided into four parts, three of these are to be divided into nine parts, and then four of these nine are to be taken; thus—

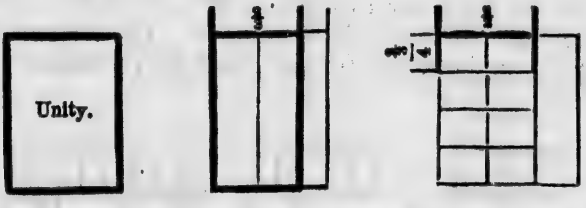


Note.—The word "of," placed between the several parts of a compound fraction, is equal to and may be replaced by ×, the sign of multiplication.

21. A Complex Fraction is one having a fraction or a mixed number in its numerator or denominator, or in both.

Thus, $\frac{2}{\frac{3}{5}}$, $\frac{\frac{4}{11}}{7}$, $\frac{8}{\frac{7}{11}}$, $\frac{\frac{3}{5}}{\frac{2}{11}}$, $\frac{9\frac{1}{2}}{18\frac{2}{13}}$, $\frac{\frac{1}{2}}{21\frac{1}{3}}$, $\frac{\frac{6\frac{1}{4}}{7\frac{1}{2}}}{\frac{3}{5\frac{1}{3}}}$, &c., are complex fractions.

Note. $\frac{\frac{2}{3}}{4}$ means, that we are to take the fourth part, not of unity, but of the $\frac{2}{3}$ of unity. This will be exemplified by—



22. Since fractions, like integers, are capable of being increased or diminished, they may be added, subtracted, &c.

23. Every integer may be considered as a fraction having *unity* for its denominator.

Thus, 13 may be written $\frac{13}{1}$; 6, $\frac{6}{1}$; 29, $\frac{29}{1}$, &c.

REDUCTION OF FRACTIONS.

24. Since (Art. 8) multiplying both numerator and denominator by the same number does not alter the value of the fraction, we may reduce an integer to a fraction having any proposed denominator, by the following :—

RULE.

Write the integral number in the form of a fraction having 1 for its denominator. (Art. 23.)

And multiply both numerator and denominator of the resulting expression by the proposed denominator. (Art. 8.)

Example 1.—Reduce 16 to a fraction having 11 for its denominator.

$$16=\tfrac{16}{1}\times\tfrac{11}{11}=\tfrac{176}{11}$$

Example 2.—Reduce 173 to a fraction having 31 for its denominator.

$$173=\tfrac{173}{1}\times\tfrac{31}{31}=\tfrac{5363}{31}.$$

Exercise 41.

1. Reduce 29 to a fraction having 12 for its denominator. *Ans.* $\frac{348}{12}$.
2. Reduce 243 to a fraction having 3 for its denominator. *Ans.* $\frac{729}{3}$.
3. Reduce 7, 23, and 101 to fractions having 13 for denominator. *Ans.* $\frac{91}{13}$, $\frac{299}{13}$, $\frac{1313}{13}$.
4. Reduce 4, 37, 126, 73, and 1007 to fractions having 101 for denominator.
5. Reduce 204, 7011, and 1999 to fractions having 207 for denominator.

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25. Let it be required to reduce the mixed number $8\frac{7}{11}$ to an improper fraction.

$8\frac{7}{11}$ is equal to the whole number 8, and the fraction $\frac{7}{11}$, and by (Art. 24.) $8=\frac{88}{11}$, therefore $8\frac{7}{11}=\frac{88}{11}+\frac{7}{11}=\frac{95}{11}$.

Hence, to reduce a mixed number to an improper fraction, we deduce the following—

RULE.

Multiplying the whole number by the denominator of the fraction, to the product add the given numerator and place the sum over the given denominator.

Example 1.—Reduce $73\frac{4}{9}$ to an improper fraction.

OPERATION.	
$73\frac{4}{9}$ 9 $\frac{661}{9}$ *Ans.*	**Explanation.**—We multiply the whole number, 73, by 9 and add in the numerator, 4. This gives us 661, which we write over the given denominator, 9, and the resulting fraction, $\frac{661}{9}$, is the improper fraction sought.

Example 2.—Reduce $276\frac{17}{20}$ to an improper fraction.

$$276\tfrac{17}{20}=\tfrac{276\times20+17}{20}=\tfrac{5537}{20}\ \textit{Ans.}$$

EXERCISE 42.

1. Reduce the mixed numbers, $78\frac{11}{15}$, $18\frac{4}{11}$, and $128\frac{1}{8}$ to improper fractions. *Ans.* $\frac{1181}{15}$, $\frac{202}{11}$, and $\frac{1025}{8}$.
2. Reduce the mixed numbers $384\frac{5}{9}$, $673\frac{2}{13}$, $4792\frac{1}{25}$, and $568\frac{2}{29}$ to improper fractions. *Ans.* $\frac{3461}{9}$, $\frac{8751}{13}$, $\frac{119801}{25}$, and $\frac{16474}{29}$.

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26. Since every fraction indicates the division of the numerator by the denominator—to reduce an improper fraction to a mixed number, we have the following—

RULE.

Divide the numerator by the denominator and the quotient will be the required mixed number.

EXAMPLE 1.—Reduce $\frac{204}{7}$ to a mixed number.

$\frac{204}{7}=204\div7=29\frac{1}{7}$ *Ans.*

EXAMPLE 2.—Reduce $\frac{20047}{11}$ to a mixed number.

$20047\div11=1822\frac{5}{11}$ *Ans.*

EXERCISE 43.

1. Reduce the improper fractions $\frac{407}{13}$, $\frac{2029}{43}$, and $\frac{19476}{1217}$ to mixed numbers. *Ans.* $31\frac{4}{13}$, $47\frac{8}{43}$, and $16\frac{4}{1217}$.
2. Reduce the improper fractions $\frac{2817}{32}$, $\frac{3964}{25}$, and $\frac{2964}{38}$ to mixed numbers. *Ans.* $88\frac{1}{32}$, $158\frac{14}{25}$, and 78.

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27. To reduce a fraction to its lowest terms—

RULE.

Divide both terms by their greatest common measure.

This is simply dividing both terms by the same number—which does not affect the value of the fraction. (Art. 8.)

The greatest common measure may be found by (Art. 26, Sec. III.) or, very frequently, by inspection.

EXAMPLE 1.—Reduce $\frac{50}{75}$ to its lowest terms.

Greatest common measure=25. Dividing both terms by 25; $\frac{50}{75}=\frac{2}{3}$ *Ans.*

EXAMPLE 2.—Reduce $\frac{126}{162}$ to its lowest terms.

Greatest common measure of 126 and 162=18.
Dividing both terms by 18 we get $\frac{126}{162}=\frac{7}{9}$ *Ans.*

EXERCISE 44.

1. Reduce $\frac{29}{3190}$ to its lowest terms. *Ans.* $\frac{1}{110}$.
2. Reduce $\frac{27378}{17370}$ to its lowest terms. *Ans.* $\frac{1521}{965}$.

3. Reduce [illegible] and [illegible] to their lowest terms. *Ans.* [illegible] and [illegible].
4. Reduce [illegible], $\frac{512}{1728}$ and [illegible] to their lowest terms.

Ans. [illegible], $\frac{8}{27}$, and [illegible].

28. *Instead of dividing both terms by their greatest common measure we may divide both by any common measure. We thus reduce the fraction to lower terms, and, continuing the division as long as the terms have a common measure, we shall finally have reduced the fraction to its lowest terms.*

NOTE.—It is advisable to commit to memory the properties of numbers given in Art. 19, Sec. III, from XVIII to XXIV.

EXAMPLE 1.—Reduce $\frac{222480}{345600}$ to its lowest terms.

$\frac{222480}{345600}$ dividing by 10. (XXI. of Art. 19, Sec. III.)
$=\frac{22248}{34560}$ dividing by 8. (XXIII. of Art. 19, Sec. III.)
$=\frac{2781}{4320}$ dividing by 9. (XXIV. of Art. 19, Sec. III.)
$=\frac{309}{480}$ dividing by 3. (XXV. of Art. 19, Sec. III.)
$=\frac{103}{160}$ *Ans.*

EXAMPLE 2.—Reduce $\frac{2835}{3915}$ to its lowest terms.

$\frac{2835}{3915}$ dividing by 5. (XX. in Art. 19, Sec. III.)
$=\frac{567}{783}$ dividing by 9. (XXIV. in Art. 19, Sec. III.)
$=\frac{63}{87}$ dividing by 3. (XXV. in Art. 19, Sec. III.)
$=\frac{21}{29}$ *Ans.*

EXERCISE 45.

1. Reduce $\frac{204}{780}$ to its lowest terms. *Ans.* $\frac{17}{65}$.
2. Reduce $\frac{5355}{136800}$ to its lowest terms. *Ans.* $\frac{119}{3040}$.
3. Reduce $\frac{2304000}{5376000}$ to its lowest terms. *Ans.* $\frac{3}{7}$.
4. Reduce $\frac{1134}{15480}$ to its lowest terms. *Ans.* $\frac{63}{860}$.
5. Reduce $\frac{88}{968}$, $\frac{549}{7143}$ and $\frac{16290}{27000}$ to their lowest terms.

Ans. $\frac{1}{11}$, $\frac{183}{2381}$, and $\frac{181}{300}$.

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29. To reduce fractions of different denominators to equivalent fractions having the same denominator—

RULE.

Multiply each numerator by all the denominators except its own for a new numerator, and all the denominators together for a new denominator.

This is merely multiplying both numerator and denominator of each fraction by the same quantity, viz: the product of all the other denominators, and consequently (Art. 8.) it does not alter the value of the fraction.

EXAMPLE 1.—Reduce $\frac{3}{4}$, $\frac{7}{11}$ and $\frac{5}{9}$ to a common denominator.

$3\times11\times\ 9=297=$1st numerator.
$7\times\ 4\times\ 9=252=$2nd numerator.
$5\times\ 4\times11=220=$3rd numerator.
$4\times11\times\ 9=396=$common denominator.

Therefore the equivalent fractions are $\frac{297}{396}$, $\frac{252}{396}$, and $\frac{220}{396}$.

Example 2.—Reduce $\frac{1}{2}$, $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{9}{11}$ to equivalent fractions having a common denominator.

$1 \times 5 \times 7 \times 11 = 385 =$ 1st numerator.
$3 \times 2 \times 7 \times 11 = 462 =$ 2nd numerator.
$4 \times 2 \times 5 \times 21 = 440 =$ 3rd numerator.
$9 \times 2 \times 5 \times 7 = 630 =$ 4th numerator.
$2 \times 5 \times 7 \times 11 = 770 =$ common denominator.

And the equivalent fractions are $\frac{385}{770}$, $\frac{462}{770}$, $\frac{440}{770}$ and $\frac{630}{770}$.

Exercise 46.

1. Reduce $\frac{2}{5}$, $\frac{5}{7}$, $\frac{8}{9}$, $\frac{3}{5}$, and $\frac{5}{18}$ to equivalent fractions having a common denominator. *Ans.* $\frac{11340}{28350}$, $\frac{20250}{28350}$, $\frac{25200}{28350}$, $\frac{17010}{28350}$, $\frac{7875}{28350}$.
2. Reduce $\frac{8}{11}$, $\frac{12}{13}$, and $\frac{5}{14}$ to fractions having a common denominator. *Ans.* $\frac{1456}{2002}$, $\frac{1848}{2002}$, $\frac{715}{2002}$.
3. Reduce $\frac{6}{7}$, $\frac{4}{11}$, $\frac{5}{13}$, $\frac{4}{7}$ and $\frac{1}{2}$ to fractions having a common denominator. *Ans.* $\frac{12012}{14014}$, $\frac{5096}{14014}$, $\frac{5390}{14014}$, $\frac{8008}{14014}$, and $\frac{7007}{14014}$.
4. Reduce $\frac{6}{11}$, $\frac{4}{7}$, and $\frac{8}{13}$ to a common denominator. *Ans.* $\frac{546}{1001}$, $\frac{572}{1001}$, and $\frac{616}{1001}$.
5. Reduce $\frac{5}{6}$, $\frac{4}{7}$, $\frac{4}{5}$, and $\frac{2}{11}$ to a common denominator. *Ans.* $\frac{1925}{2310}$, $\frac{1320}{2310}$, $\frac{1848}{2310}$, and $\frac{420}{2310}$.
6. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{2}{7}$ to a common denominator. *Ans.* $\frac{105}{210}$, $\frac{140}{210}$, $\frac{126}{210}$, and $\frac{60}{210}$.

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30. To reduce fractions to equivalent fractions having their *least* common denominator—

RULE.

Find the least common multiple of all the denominators. (*Art. 33, Sec. III.*)

Multiply both terms of each fraction by the quotient obtained by dividing this least common multiple by the denominator of that fraction.

This is merely multiplying both terms by the same quantity, as in Art. 29.

Example 1.—Reduce $\frac{1}{4}$, $\frac{7}{12}$, $\frac{2}{3}$, and $\frac{9}{16}$ to their least common denominator.

The least common multiple of 4, 12, 3, and 16, is 48.
Multiplying both terms of the 1st fraction by 12 (i. e. $\frac{48}{4}$) it becomes $\frac{12}{48}$.
" " " 2nd " by 4 (i. e. $\frac{48}{12}$) it becomes $\frac{28}{48}$.
" " " 3rd " by 16 (i. e $\frac{48}{3}$) it becomes $\frac{32}{48}$.
" " " 4th " by 3 (i. e. $\frac{48}{16}$) it becomes $\frac{27}{48}$.

The equivalent fractions having their *least* common denominator, are therefore $\frac{12}{48}$, $\frac{28}{48}$, $\frac{32}{48}$, and $\frac{27}{48}$.

EXAMPLE 2.—Reduce $\frac{4}{5}$, $\frac{6}{11}$, $\frac{13}{20}$, $\frac{21}{44}$, $\frac{18}{55}$, and $\frac{3}{4}$ to their least common denominator.

The least common multiple of 5, 11, 20, 44, 55, and 4, is 220.

The multiplier for both terms of the first fraction is $\frac{220}{5}=44$, for second, $\frac{220}{11}=20$; for the third, $\frac{220}{20}=11$; for the fourth, $\frac{220}{44}=5$; for the fifth, $\frac{220}{55}=4$; and for the sixth, $\frac{220}{4}=55$.

Multiplying by these numbers, we obtain $\frac{176}{220}$, $\frac{120}{220}$, $\frac{143}{220}$, $\frac{105}{220}$, $\frac{72}{220}$, and $\frac{165}{220}$ for the required fractions.

EXERCISE 47.

1. Reduce $\frac{1}{6}$, $\frac{3}{8}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{7}{15}$ to their least common denominator. *Ans.* $\frac{20}{120}$, $\frac{45}{120}$, $\frac{80}{120}$, $\frac{90}{120}$, and $\frac{56}{120}$.
2. Reduce $\frac{6}{11}$, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{18}{77}$, and $\frac{19}{33}$ to their least common denominator. *Ans.* $\frac{126}{231}$, $\frac{154}{231}$, $\frac{132}{231}$, $\frac{54}{231}$, and $\frac{133}{231}$.
3. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{19}{20}$, $\frac{13}{16}$, and $\frac{37}{40}$ to their least common denominator. *Ans.* $\frac{120}{240}$, $\frac{160}{240}$, $\frac{180}{240}$, $\frac{144}{240}$, $\frac{200}{240}$, $\frac{210}{240}$, $\frac{216}{240}$, $\frac{228}{240}$, $\frac{195}{240}$, and $\frac{222}{240}$.
4. Reduce $\frac{3}{5}$, $\frac{7}{15}$, $\frac{6}{25}$, $\frac{11}{30}$, $\frac{13}{45}$, and $\frac{23}{60}$ to their least common denominator. *Ans.* $\frac{540}{900}$, $\frac{420}{900}$, $\frac{216}{900}$, $\frac{330}{900}$, $\frac{260}{900}$, and $\frac{345}{900}$.
5. Reduce $\frac{19}{20}$, $\frac{7}{30}$, $\frac{11}{40}$, and $\frac{1}{50}$ to their least common denominator. *Ans.* $\frac{570}{600}$, $\frac{140}{600}$, $\frac{165}{600}$, and $\frac{12}{600}$.
6. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{15}{16}$, and $\frac{23}{24}$ to their least common denominator. *Ans.* $\frac{24}{48}$, $\frac{32}{48}$, $\frac{36}{48}$, $\frac{40}{48}$, $\frac{42}{48}$, $\frac{44}{48}$, $\frac{45}{48}$, and $\frac{46}{48}$.
7. Reduce $\frac{5}{7}$, $\frac{11}{12}$, $\frac{2}{15}$, $\frac{8}{27}$, $\frac{9}{35}$, and $\frac{17}{40}$ to their least common denominator. *Ans.* $\frac{5400}{7560}$, $\frac{6930}{7560}$, $\frac{1008}{7560}$, $\frac{2240}{7560}$, $\frac{1944}{7560}$, and $\frac{3213}{7560}$.
8. Reduce $\frac{14}{15}$, $\frac{7}{8}$, $\frac{4}{3}$, $\frac{11}{12}$, $\frac{6}{11}$, $\frac{19}{20}$, $\frac{6}{7}$, and $\frac{29}{35}$ to their least common denominator. *Ans.* $\frac{8624}{9240}$, $\frac{8085}{9240}$, $\frac{12320}{9240}$, $\frac{8470}{9240}$, $\frac{5040}{9240}$, $\frac{8778}{9240}$, $\frac{7920}{9240}$, and $\frac{7656}{9240}$.

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31. Let it be required to reduce $\frac{12}{17}$ of $\frac{6}{11}$ to a simple fraction.

$\frac{12}{17}$ of $\frac{6}{11}$ means 12 times $\frac{1}{17}$ of $\frac{6}{11}$.

We get $\frac{1}{17}$ of $\frac{6}{11}$, i. e. divide $\frac{6}{11}$ by 17, when we multiply the denominator 11 by 17 (Art. 8). Therefore $\frac{1}{17}$ of $\frac{6}{11}=\frac{6}{11\times17}$, and to multiply this result by 12, we multiply the numerator, 6, by 12, (Art. 8.)

Therefore $\frac{12}{17}$ of $\frac{6}{11}=\frac{6\times12}{11\times17}=\frac{72}{187}$.

Hence to reduce a compound fraction to a simple one we deduce the following—

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

EXAMPLE.—Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $\frac{5}{9}$ to a simple fraction.

$$\tfrac{2}{3}\text{ of }\tfrac{4}{7}\text{ of }\tfrac{5}{9}=\frac{2\times4\times5}{3\times7\times9}=\tfrac{40}{189}\ \textit{Ans.}$$

NOTE.—In all cases the answer must be reduced to its lowest terms.

EXERCISE 48.

1. Reduce $\frac{4}{7}$ of $\frac{3}{5}$ of $\frac{6}{11}$ of $\frac{35}{72}$ to a simple fraction. *Ans.* $\frac{1}{11}$.
2. Reduce $\frac{8}{9}$ of $\frac{1}{3}$ of $\frac{6}{7}$ of $\frac{81}{100}$ of $\frac{25}{24}$ to a simple fraction. *Ans.* $\frac{3}{14}$.
3. Reduce $\frac{21}{35}$ of $\frac{6}{11}$ of $\frac{77}{36}$ to a simple fraction. *Ans.* $\frac{7}{10}$.
4. Reduce $\frac{2}{5}$ of $\frac{4}{7}$ of $\frac{3}{11}$ of $\frac{13}{17}$ to a simple fraction. *Ans.* $\frac{312}{6545}$.

32. Since the several numerators of the compound fraction form the factors of the numerator of the simple fraction, and also the several denominators of the compound fraction, the factors of the denominator of the simple fraction, it follows (Art. 8.) that,—

Before applying the rule in (Art. 31) *we may cast out or cancel all the factors that are common to a numerator and a denominator of the compound fraction.*

EXAMPLE 1.—Reduce $\frac{6}{11}$ of $\frac{4}{7}$ of $\frac{3}{5}$ of $\frac{22}{27}$ of $\frac{35}{16}$ to a simple fraction.

STATEMENT. CANCELLED.

$$\frac{6}{11}\text{ of }\frac{4}{7}\text{ of }\frac{3}{5}\text{ of }\frac{22}{27}\text{ of }\frac{35}{16}=\frac{6\times4\times3\times22\times35}{11\times7\times5\times27\times16}=\frac{\overset{2}{\cancel{6}}\times\cancel{4}\times\cancel{3}\times\overset{2}{\cancel{22}}\times\overset{5}{\cancel{35}}}{\cancel{11}\times\cancel{7}\times\cancel{5}\times\underset{\substack{\cancel{9}\\3}}{\cancel{27}}\times\underset{\substack{\cancel{8}\\\cancel{4}}}{\cancel{16}}}=\frac{1}{3}\ \textit{Ans.}$$

Here 6 and 27 contain a common factor, 3, which is cast out, and these numbers thus reduced to 2 and 9. Next this 2 reduces 16 to 8, and the 9 is reduced to 3 by the third numerator, which is thus cancelled. Again, 11 cancels 11 (the first denominator) and reduces 22 to 2, and this 2 reduces the 8, before obtained from the 16, to 4. Next, this 4 is cancelled by the 4 in the numerator. Again, 7 cancels the 7 in the denominator and reduces the 35, in the numerator, to 5, and this 5 cancels the 5 in the denominator. All the numerators are now reduced to unity, as also all the denominators but the fourth, which is 3. The resulting fraction is therefore $\frac{1\times1\times1\times1\times1}{1\times1\times1\times3\times1}$ but this is simply $\frac{1}{3}$.

EXAMPLE 2.—Reduce $\frac{7}{11}$ of $\frac{4}{6}$ of $\frac{3}{5}$ of $\frac{55}{20}$ to a simple fraction.

STATEMENT. CANCELLED.

$$\frac{7}{11}\text{ of }\frac{4}{6}\text{ of }\frac{3}{5}\text{ of }\frac{55}{20}=\frac{7\times4\times3\times55}{11\times6\times5\times20}=\frac{7\times\cancel{4}\times\cancel{3}\times\overset{\cancel{5}}{\cancel{55}}}{\cancel{11}\times\underset{2}{\cancel{6}}\times\cancel{5}\times\underset{5}{\cancel{20}}}=\frac{7}{2\times5}=\frac{7}{10}\ \textit{Ans.}$$

NOTE.—If any of the terms of the compound fraction are whole or mixed numbers, they must be reduced to fractions (Arts. 28 and 25.)

The process of cancelling exemplified above should always be adopted when possible.

EXERCISE 40.

1. Reduce $\frac{5}{8}$ of $\frac{4}{7}$ of $\frac{2}{3}$ of $\frac{3}{10}$ to a simple fraction. *Ans.* $\frac{1}{14}$.
2. Reduce $\frac{2}{3}$ of $\frac{5}{8}$ of $\frac{14}{153}$ of $\frac{6}{11}$ of $\frac{11}{13}$ of $\frac{14}{17}$ to a simple fraction. *Ans.* $\frac{10}{561}$.
3. Reduce $\frac{3}{8}$ of $\frac{4}{11}$ of $5\frac{1}{2}$ to a simple fraction. *Ans.* $\frac{3}{4}$.
4. Reduce $\frac{1}{3}$ of $\frac{8}{15}$ of $\frac{117}{200}$ of $\frac{50}{169}$ of $\frac{14}{17}$ of $2\frac{1}{3}$ to a simple fraction. *Ans.* $\frac{1}{45}$.
5. Reduce $\frac{3}{11}$ of $\frac{4}{7}$ of $\frac{9}{16}$ of $2\frac{4}{7}$ of $\frac{7}{8}$ of $6\frac{2}{7}$ to a simple fraction. *Ans.* $\frac{3}{25}$.
6. Reduce $\frac{4}{7}$ of $\frac{3}{11}$ of 154 to a simple fraction. *Ans.* 24.

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33. Let it be required to reduce the complex fraction $\frac{\frac{6}{7}}{\frac{3}{4}}$ to a simple fraction.

Since (Art. 8) we may multiply both numerator and denominator of a fraction by the same number, without altering its value—we may multiply both terms of the given fraction by $\frac{4}{3}$, i. e., by the denominator with its terms inverted, without altering its value.

$$\text{Therefore } \frac{\frac{6}{7}}{\frac{3}{4}} = \frac{\frac{6}{7} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{6}{7} \times \frac{4}{3}}{1} = \frac{6}{7} \times \frac{4}{3} = \frac{6 \times 4}{7 \times 3}.$$

Hence to reduce a complex fraction to a simple one, we deduce the following:—

RULE.

Reduce the expression (Arts. 28 and 25) to the form of $\frac{\textit{fraction}}{\textit{fraction}}$; *i. e., reduce both numerator and denominator to simple fractions. Then multiply the extremes or outside numbers together for a new numerator, and the means or intermediate numbers together for a new denominator.*

EXAMPLE 1.—Reduce $\frac{4\frac{1}{2}}{\frac{7}{11}}$ to a simple fraction.

$$\frac{4\frac{1}{2}}{\frac{7}{11}} = \frac{\frac{9}{2}}{\frac{7}{11}} = \frac{9 \times 11}{2 \times 7} = \frac{99}{14} = 7\tfrac{1}{14}\ \textit{Ans.}$$

NOTE.—Factors that are common to one of the extremes and one of the means, are to be struck out or cancelled. (Art. 32).

EXAMPLE 2.—Reduce $\frac{7\frac{4}{11}}{1\frac{13}{77}}$ to a simple fraction.

$$\frac{7\frac{4}{11}}{1\frac{13}{77}} = \frac{\;\frac{\overset{9}{\cancel{81}}}{\cancel{11}}\;}{\;\frac{\cancel{90}\ 10}{\underset{7}{\cancel{77}}}\;} = \frac{7 \times 9}{10} = \frac{63}{10} = 6\tfrac{3}{10}\ \textit{Ans.}$$

Exercise 50.

1. Reduce $\frac{\frac{1}{4}\frac{5}{8}}{1\frac{1}{2}\frac{7}{8}}$ to a simple fraction. *Ans.* $\frac{5}{42}$.

2. Reduce $\frac{\frac{1}{4}\frac{1}{2}}{7\frac{1}{7}\frac{7}{8}}$ to a simple fraction. *Ans.* $\frac{2}{35}$.

3. Reduce $\frac{15\frac{3}{5}}{7\frac{4}{5}}$ to a simple fraction. *Ans.* 2.

4. Reduce $\frac{11\frac{2}{3}}{12\frac{5}{8}}$, $\frac{3\frac{1}{4}}{9}$ and $\frac{\frac{2}{7}}{\frac{3}{5}}$ to simple fractions.
Ans. $\frac{175}{204}$, $\frac{13}{36}$, and $\frac{10}{21}$.

5. Reduce $\frac{\frac{7}{12}}{15\frac{3}{4}}$, $\frac{5\frac{7}{8}}{\frac{3}{16}}$ and $\frac{2\frac{2}{5}}{3\frac{3}{7}}$ to simple fractions.
Ans. $\frac{1}{27}$, $31\frac{1}{3}$, and $\frac{7}{10}$.

6. Reduce $\frac{16\frac{2}{3}}{11\frac{2}{3}}$, $\frac{6\frac{1}{5}}{13}$, $\frac{17}{18\frac{1}{3}}$, $\frac{21\frac{3}{5}}{10\frac{2}{7}}$ and $\frac{\frac{1}{2}}{4\frac{3}{5}}$ to simple fractions.
Ans. $1\frac{3}{7}$, $\frac{31}{65}$, $\frac{51}{55}$, $2\frac{1}{10}$, and $\frac{5}{46}$.

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34. A denominate fraction is a fraction of a denominate number.

Thus, $\frac{3}{4}$ of a lb., $\frac{1}{11}$ of a mile, $\frac{3}{5}$ of a day, &c., are denominate fractions.

35. Reduction of denominate fractions consists in changing them from one denomination to another without altering their values.

36. Let it be required to reduce $\frac{4}{7}$ of a pint to the fraction of a bushel.

Since 1 qt. = 2 pints, $\frac{4}{7}$ of a pint = $\frac{1}{2}$ of $\frac{4}{7}$ of a quart.
Also because 1 gal. = 4 qts. $\frac{4}{7}$ of a pint = $\frac{1}{4}$ of $\frac{1}{2}$ of $\frac{4}{7}$ of a gal.
Similarly $\frac{4}{7}$ of a pint = $\frac{1}{4}$ of $\frac{1}{2}$ of $\frac{1}{4}$ of $\frac{1}{2}$ of $\frac{4}{7}$ of a bushel = $\frac{4}{448}$ = $\frac{1}{112}$ bushel.

Hence to reduce a denominate fraction from a lower to a higher denomination, we deduce the following:—

RULE.

Take the number expressing how many of the given denomination are required to make one of the next higher; also the number expressing how many of this denomination are required to make one of the next higher again, and so on until the required denomination be reached.

Write the fractions formed by these numbers as denominators, with 1 as numerator and the given fraction in the form of a compound fraction, which reduce to a simple fraction. (*Art.* 31.)

denominata

Example 1.—Reduce $\frac{3}{11}$ of a minute to the fraction of a week.
Ans. $\frac{3}{11}$ of $\frac{1}{60}$ of $\frac{1}{24}$ of $\frac{1}{7}=\frac{1}{36960}$ of a week.

Example 2.—Reduce $\frac{64}{65}$ of a grain troy, to the fraction of an ounce.

$\frac{64}{65}$ of $\frac{1}{24}$ of $\frac{1}{20}=\frac{2}{975}$ of an oz. Troy.

Exercise 51.

1. Reduce $\frac{4}{5}$ of an oz. to the fraction of a pound, avoirdupois. *Ans.* $\frac{1}{20}$ lb.
2. Reduce $\frac{2}{3}$ of $\frac{3}{7}$ of a penny to the fraction of a pound. *Ans.* £$\frac{1}{840}$.
3. Reduce $\frac{2}{9}$ of $8\frac{3}{4}$ days to the fraction of a week. *Ans.* $\frac{5}{18}$ wk.
4. Reduce $\frac{5}{11}$ of $16\frac{1}{5}$ nails to the fraction of an English ell. *Ans.* $\frac{81}{220}$ E. e.
5. Reduce $\frac{3}{7}$ of $\frac{4}{11}$ of a yard to the fraction of a perch. *Ans.* $\frac{24}{847}$ per.
6. Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $21\frac{1}{14}$ of a cord foot to the fraction of a cord. *Ans.* $1\frac{1}{294}$ cord.
7. Reduce $\frac{3}{19}$ of $\frac{4}{17}$ of $9\frac{1}{2}$ square perches to the fraction of an acre. *Ans.* $\frac{3}{1360}$ acre.

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37. Let it be required to reduce $\frac{4}{5}$ of a day to the fraction of a minute.

Since there are 24 hours in a day and 60 minutes in an hour;

$\frac{4}{5}$ of a day will be 24 times $\frac{4}{5}$ of an hour and 60 times 24 times $\frac{4}{5}$ of a minute; that is, $\frac{4}{5}$ of a day is equal to $\frac{4}{5}\times 24\times 60$ of a minute.

Therefore $\frac{4}{5}$ of a day $=\frac{4}{5}$ of $\frac{24}{1}$ of $\frac{60}{1}$ of a minute $=\frac{1152}{1}$ minute.

Hence, to reduce a denominate fraction from a higher to a lower denomination, we have the following—

RULE.

Take the number expressing how many of the next lower denomination make one of the given denomination; also, the number, expressing how many of the next lower again make one of this denomination, and so on till the required denomination be reached.

Write the fractions formed by these numbers as numerators, with 1 as denominator, as the given fraction in the form of a compound fraction, which reduce to a simple fraction. (*Art.* 31.)

Example 1.—Reduce $\frac{2}{3}$ of a £ to the fraction of a penny.

$\frac{2}{3}$ of $\frac{20}{1}$ of $\frac{12}{1}=\frac{480}{3}=160$ pence.

Example 2.—Reduce $\frac{4}{3}$ of $\frac{5}{6}$ of $\frac{12}{11}$ of a furlong to the fraction of a foot.

$\frac{4}{3}$ of $\frac{5}{6}$ of $\frac{12}{11}$ of $\frac{40}{1}$ of $\frac{11}{2}$ of $\frac{3}{1}=800$ ft. *Ans.*

EXERCISE 52.

1. Reduce $\frac{14}{15}$ of a bushel to the fraction of a quart. *Ans.* $\frac{448}{15}$ qt.
2. Reduce $\frac{3}{5}$ of a gallon to the fraction of $\frac{1}{5}$ of $\frac{3}{8}$ of a gill. *Ans.* $\frac{160}{3}$.
3. Reduce $\frac{7}{9}$ of 2 pecks to the fraction of $\frac{1}{2}$ of $\frac{2}{3}$ of a pint. *Ans.* $\frac{224}{3}$.
4. Reduce $\frac{17}{22}$ of a pound to the fraction of a scruple. *Ans.* $\frac{2448}{11}$ scr.
5. Reduce $\frac{1}{5000}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{6}{11}$ of $\frac{22}{7}$ of a lb. avoirdupois to the fraction of a dram. *Ans.* $\frac{192}{4375}$ dr.

38. To find the value of a denominate fraction in terms of a lower denomination—

RULE.

Divide the numerator by the denominator according to the rule given in Art. 71, *Sec. II.*

This is only actually performing the work which the fraction indicates. (Art. 3.)

EXAMPLE.—What is the value of $\frac{11}{13}$ of a mile?

11 miles ÷ 13

13)11 miles (6 fur. 30 per. $4\frac{3}{13}$ yds. *Ans.*
8=fur. in a mile.
—
88=number of furlongs.
78
—
10
40=perches in furlong.
—
400=perches.
390
—
10
$5\frac{1}{2}$=yards in a perch.
—
55=number of yards.
52
—
3

EXERCISE 53.

1. What is the value of $\frac{3}{11}$ of a bushel and also of $\frac{6}{7}$ of a lb. avoirdupois? *Ans.* 1 pk. 0 gal. 0 qt. $1\frac{5}{11}$ pt. and 13 oz. $11\frac{3}{7}$ drams.
2. What is the value of $\frac{7}{13}$ of a yard of cloth? *Ans.* 2 qrs. 0 na. $1\frac{5}{13}$ inches.
3. What is the value of $\frac{8}{9}$ of a lb. troy; and also of $\frac{11}{113}$ sq. mile? *Ans.* 10 oz. 13 dwt. 8 grs.; and 62 acres, 1 rood, 8 sq. per. 4 sq. yds. 2 ft. $79\frac{1}{113}$ in.

4. What is the value of $\frac{8}{9}$ of a furlong; and of $\frac{4}{7}$ of a £?
Ans. 35 rds. 3 yds. 0 ft. 2 in.; and 11s. $5\frac{1}{7}$d.

---

39. Let it be required to reduce 2s. $7\frac{3}{4}$d. to the fraction of £7 18s.

$$\frac{\text{2s. }7\frac{3}{4}\text{d}}{\text{£7 18s.}}=\frac{127\text{ farthings.}}{7584\text{ farthings.}}\text{ Therefore 2s. }7\tfrac{3}{4}\text{d.}=\frac{127}{7584}\text{ of £7 18.}$$

Hence, to reduce one denominate number to the fraction of another, we deduce the following—

RULE.

Reduce both quantities to the lowest denomination contained in either.

Then place that quantity which is to be the fraction of the other as numerator and the remaining quantity as denominator.

Example 1.—Reduce 3 days 4 hours to the fraction of a week.

3 days 4 hours=76 hours.
1 week=168 hours.
And the required fraction is $\frac{76}{168}=\frac{19}{42}$ *Ans.*

Example 2.—What fraction is 3 lb. 4 oz. 2 dr. 2 scr. 7 grs. of 63 lb. 4 oz. 7 dr. Apothecaries' weight?

3 lb. 4 oz. 2 dr. 2 scr. 7 grs.=19367 grs.
63 lb. 4 oz. 7 dr.=365220 grs.
And the fraction is $\frac{19367}{365220}$ *Ans.*

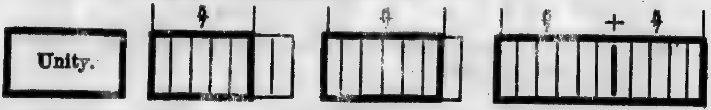
Exercise 54.

1. What fraction is 6 bush. 1 pk. 1 gal. 1 qt. 1 pt. of 50 bush.? *Ans.* $\frac{411}{3200}$.
2. What fraction is 35 per. 9 ft. 2 in. of a furlong? *Ans.* $\frac{8}{9}$.
3. What fraction is 7 h. 12 m. of a day? *Ans.* $\frac{3}{10}$.
4. What fraction is 2 sq. yds. 2 ft. 120 in. of 3 sq. per. $13\frac{1}{4}$ yds. 1 ft. 72 in.? *Ans.* $\frac{1}{45}$.
5. What fraction is 7 oz. 7 dr. 2 scr. 14 grs. of 21 lbs. Apoth.? *Ans.* $\frac{71}{2240}$.
6. Reduce 9 min. 48 sec. to the fraction of a day. *Ans.* $\frac{49}{7200}$.
7. Reduce 16 bush. 1 pk. 1 pt. to the fraction of 69 bush. *Ans.* $\frac{347}{1472}$.
8. Reduce 3 qrs. $3\frac{1}{3}$ na. to the fraction of an ell Eng. *Ans.* $\frac{23}{30}$.
9. What part of a lb. Troy is 13 dwt. 7 grs.? *Ans.* $\frac{319}{5760}$.
10. What part of 54 cords of wood is 4800 cubic feet? *Ans.* $\frac{25}{36}$.

ADDITION OF VULGAR FRACTIONS.

40. Addition of fractions is the process of finding a single fraction which shall express the value of all the fractions added.

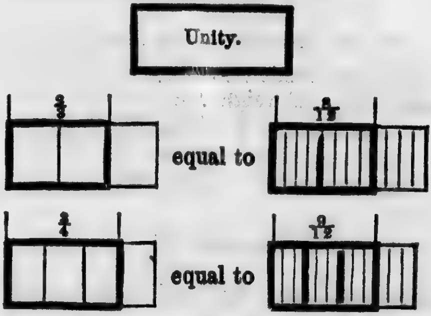
Addition may be illustrated as follows :—



41. In order that fractions may be added they must have a common denominator.

Thus $\frac{2}{3}+\frac{3}{4}$ make neither $\frac{5}{7}$ nor $\frac{5}{12}$; but if we reduce them to equivalent fractions having a common denominator, as $\frac{8}{12}$ and $\frac{9}{12}$, we are enabled to add them and thus obtain for their sum $\frac{17}{12}$.

These fractions, before and after they receive a common denominator, will be represented as follows :—



We have increased the number of the parts just as much as we have diminished their size.

42. For the addition of fractions we have therefore the following :—

RULE.

Reduce compound and complex fractions to simple ones, and all to a common denominator. (*Arts.* 29, 31, *and* 33.)

Add all the numerators together, and beneath their sum place the common denominator.

Reduce the resulting fraction, when it is an improper fraction, to a mixed number. (*Art.* 26.)

NOTE.—*If mixed numbers occur among the addends, the integral portions are to be added separately and their sum added to the sum of the fractions.*

EXAMPLE 1.—Add together $\frac{4}{11}$, $\frac{8}{11}$, $\frac{2}{11}$, $\frac{7}{11}$ and $\frac{10}{11}$.

Here, since the fractions have already a common denominator, we have simply to add the numerators and place 11, the common denominator, beneath their sum.

Thus $\frac{4}{11}+\frac{8}{11}+\frac{2}{11}+\frac{7}{11}+\frac{10}{11}=\frac{4+8+2+7+10}{11}=\frac{31}{11}=2\frac{9}{11}$ *Ans.*

EXAMPLE 2.—Add together $\frac{2}{4}$, $\frac{3}{7}$, $\frac{4}{8}$, $\frac{6}{7}$ and $\frac{11}{14}$.

These fractions reduced to their least common denominator by *Art.* 30, become $\frac{28}{56}$, $\frac{24}{56}$, $\frac{28}{56}$, $\frac{48}{56}$, $\frac{44}{56}$.

And $\frac{28}{56}+\frac{24}{56}+\frac{28}{56}+\frac{48}{56}+\frac{44}{56}=\frac{28+24+28+48+44}{56}=\frac{172}{56}=\frac{43}{14}=3\frac{1}{14}$ *Ans.*

EXAMPLE 3.—Add together $\frac{3}{7}$, $\frac{4}{5}$, $\frac{9}{11}$ and $\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{8}{11}$ of $\frac{77}{96}$ of $5\frac{1}{4}$.

$\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{8}{11}$ of $\frac{77}{96}$ of $5\frac{1}{4}$ is equal to $\frac{7}{8}$ (*Art.* 31).
The fractions to be added are therefore $\frac{3}{7}+\frac{4}{5}+\frac{9}{11}+\frac{7}{8}$.
These reduced to a common denominator (*Art.* 29), become
$\frac{1320}{3080}+\frac{2464}{3080}+\frac{2520}{3080}+\frac{2695}{3080}=\frac{8999}{3080}=2\frac{2839}{3080}$ *Ans.*

EXAMPLE 4.—Add together $9\frac{1}{2}$, $11\frac{3}{4}$, $16\frac{7}{9}$, $43\frac{2}{5}$, and $\frac{4\frac{1}{2}}{7\frac{1}{5}}$.

Here the last fraction is a complex fraction and is equal to $\frac{5}{8}$.
$9\frac{1}{2}+11\frac{3}{4}+16\frac{7}{9}+43\frac{2}{5}+\frac{5}{8}=9+11+16+43+(\frac{1}{2}+\frac{3}{4}+\frac{7}{9}+\frac{2}{5}+\frac{5}{8})$.
And $9+11+16+43=79$.
Also $\frac{1}{2}+\frac{3}{4}+\frac{7}{9}+\frac{2}{5}+\frac{5}{8}=\frac{180}{360}+\frac{270}{360}+\frac{280}{360}+\frac{144}{360}+\frac{225}{360}=\frac{1099}{360}=3\frac{19}{360}$.
Therefore the sum of the given quantities is $79+3\frac{19}{360}=82\frac{19}{360}$.

EXAMPLE 5.—Add together $\frac{8}{9}$, $\frac{3}{7}$, and $5\frac{3}{8}$.

Here adding the three fractions together we obtain $1\frac{349}{504}$ for their sum, to which we add the integral number 5 and thus obtain the entire sum $6\frac{349}{504}$.

EXERCISE 55.

1. Add together $\frac{11}{13}$, $\frac{10}{13}$ and $\frac{9}{13}$. *Ans.* $\frac{30}{13}=2\frac{4}{13}$.
2. Add together $\frac{1}{12}$, $\frac{6}{12}$, $\frac{7}{12}$, $\frac{9}{12}$, $\frac{11}{12}$ and $\frac{5}{12}$.
Ans. $\frac{39}{12}=\frac{13}{4}=3\frac{1}{4}$.
3. Add together $4\frac{3}{7}$, $11\frac{4}{7}$, $16\frac{2}{7}$, $21\frac{3}{7}$ and $19\frac{6}{7}$.
Ans. $71+\frac{18}{7}=73\frac{4}{7}$.
4. Add together $16\frac{21}{23}$, $11\frac{17}{23}$, $18\frac{4}{23}$, $17\frac{19}{23}$, and $112\frac{22}{23}$.
Ans. $177\frac{14}{23}$.
5. Add together $4\frac{1}{4}$, $1\frac{1}{3}$ and $\frac{7}{11}$. *Ans.* $6\frac{29}{132}$.
6. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ and $\frac{8}{9}$. *Ans.* $6\frac{431}{2520}$.
7. Add together $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{4}{5}$. *Ans.* $2\frac{23}{60}$.
8. Add together $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{3}{8}$ and $\frac{8}{11}$. *Ans.* $3\frac{5477}{9240}$.
9. Add together $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$. *Ans.* $1\frac{83}{140}$.
10. Add together $16\frac{3}{11}$, $47\frac{2}{9}$, $21\frac{17}{33}$, $\frac{7}{18}$, and $19\frac{1}{2}$.
Ans. $104\frac{89}{99}$.
11. Add together $17\frac{1}{3}$, $43\frac{4}{7}$, $168\frac{5}{9}$, $207\frac{8}{21}$ and $506\frac{13}{63}$.
Ans. $943\frac{1}{21}$.

12. Add together $6\frac{3}{4}$, $11\frac{4}{7}$, $\frac{9}{56}$, $16\frac{7}{16}$, $\frac{1}{8}$, $\frac{5}{21}$ and $17\frac{1}{12}$. *Ans.* $53\frac{123}{336}$.

13. Add together $\frac{1}{5}$, $\frac{2}{3}$, $\frac{7}{8}$ and $68\frac{1}{4}$. *Ans.* $69\frac{119}{120}$.

14. Add together $173\frac{3}{14}$, $8\frac{3}{4}$ and $91\frac{11}{13}$. *Ans.* $273\frac{295}{364}$.

15. Add together $1\frac{15}{16}$, $2\frac{23}{24}$, $3\frac{24}{25}$ and $4\frac{29}{30}$. *Ans.* $13\frac{329}{400}$.

16. Add together $\frac{1}{6}$, $\frac{5}{12}$, $\frac{4}{45}$, $\frac{5}{24}$, $\frac{7}{16}$, $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{5}{6}$. *Ans.* $3\frac{5}{48}$.

17. Add together 7, $11\frac{1}{2}$, 18, $26\frac{3}{7}$ and $79\frac{4}{11}$. *Ans.* $142\frac{45}{154}$.

18. Add together $\frac{2}{3}$, $7\frac{2}{11}$, and $\frac{4}{5}$ of $\frac{3}{7}$ of $10\frac{1}{2}$. *Ans.* $11\frac{74}{165}$.

19. Add together $\dfrac{4\frac{1}{3}}{\frac{7}{18}}$, $\frac{1}{2}$ of $3\frac{3}{11}$ of $\frac{4}{15}$ of $2\frac{3}{4}$, and $\dfrac{20\frac{3}{4}}{7\frac{6}{11}}$. *Ans.* $15\frac{13}{140}$.

20. Add together $3\frac{5}{8}$, $11\frac{1}{6}$ and $14\frac{33}{48}$. *Ans.* $29\frac{23}{48}$.

21. Add together $\frac{1}{2}$ of $\frac{3}{4}$, $\frac{2}{3}$ of $\frac{6}{7}$, $\frac{3}{8}$ of $\frac{7}{8}$, $\frac{7}{8}$ of $1\frac{7}{20}$ and $4\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{3}$ of $\frac{1}{6}$ of $\frac{1}{3}$. *Ans.* $1\frac{1261}{1680}$.

22. Add together $41\frac{1}{2}$, $105\frac{5}{8}$, $300\frac{3}{4}$, $241\frac{3}{5}$, and $472\frac{1}{4}$. *Ans.* $1161\frac{29}{40}$.

23. Add together $92\frac{5}{14}$, $37\frac{8}{19}$ and $7\frac{2}{3}$. *Ans.* $137\frac{355}{798}$.

24. Add together $21\frac{1}{3}$, $35\frac{1}{6}$, $\dfrac{10\frac{7}{8}}{2\frac{3}{8}}$ and $\frac{2}{3}$ of $\frac{7}{8}$. *Ans.* $61\frac{5}{6}$.

25. Add together $2\frac{3}{4}$ of $3\frac{2}{3}$, $\frac{111}{116}$, $2\frac{4}{5}$ of $4\frac{1}{6}$ of $1\frac{3}{8}$, and $4\frac{2}{3}$ of $\frac{3}{13}$ of $2\frac{1}{6}$ of $1\frac{3}{7}$. *Ans.* $34\frac{1139}{1440}$.

---

43. *In order to add denominate fractions they must not only have a common denominator, but they must be fractions of the same unit, i. e., must be of the same denomination.*

Thus £$\frac{3}{4}$, $\frac{3}{4}$s. and $\frac{3}{4}$d. cannot be added together, as the result would be neither $\frac{9}{4}$ of a pound, $\frac{9}{4}$ of a shilling, nor $\frac{9}{4}$ of a penny.

But if we reduce them all to the fraction of a pound, or all to the fraction of a shilling, or all to the fraction of a penny, it is obvious that we may then add the resulting fractions, having first reduced them to a common denominator.

Hence, for the addition of denominate fractions, we have the following—

RULE.

Reduce all the fractions to the same denomination (*Arts.* 36 *and* 37). *Reduce the resulting fractions to a common denominator* (*Arts.* 29 *and* 30). *Add* (*as in Art.* 42) *and find the value of the resulting fraction* (*Art.* 38).

EXAMPLE 1.—Add together $\frac{2}{9}$ of a day and $\frac{3}{7}$ of an hour.

$\frac{2}{9}$ of a day $=\frac{2}{9}$ of $\frac{24}{1}=\frac{48}{9}=\frac{16}{3}$ of an hour.

$\frac{16}{3}$h. $+\frac{3}{7}$h. $=\frac{112}{21}+\frac{9}{21}=\frac{121}{21}=5\frac{16}{21}$h. $=$ 5h. 35m. $42\frac{6}{7}$ sec.

EXAMPLE 2.—Add together $\frac{7}{11}$ of a pound, $\frac{2}{5}$ of a shilling, and $\frac{3}{7}$ of a penny.

$\frac{7}{11}$ of a £ $=\frac{7}{11}$ of $\frac{20}{1}$ of $\frac{12}{1}=\frac{1680}{11}$ of a penny $=152\frac{8}{11}$ pence.

$\frac{2}{5}$ of a shilling $=\frac{2}{5}$ of $\frac{12}{1}=\frac{24}{5}$ of a penny $=4\frac{4}{5}$ pence.

$$152\tfrac{8}{11}+4\tfrac{4}{5}+\tfrac{3}{7}=156+\frac{280+308+165}{385}=157\tfrac{368}{385}\text{ pence}=13\text{s. }1\tfrac{368}{385}\text{d.}$$

NOTE.—In place of proceeding as above, we may find the value of each fraction separately (Art. 38) and add the results.

EXAMPLE 3.—Add together $\frac{4}{5}$ of a bushel, $\frac{7}{8}$ of a peck, and $\frac{2}{11}$ of a gallon.

$\frac{4}{5}$ of a bushel = 3 pks. 0 gal. 1 qt. $1\frac{1}{5}$ pts.
$\frac{7}{8}$ of a peck = 1 gal. 3 qts.
$\frac{2}{11}$ of a gal. = $1\frac{5}{11}$ pts.
Sum = 1 bush. 0 pks. 0 gal. 1 qt. $0\frac{36}{55}$ pts. *Ans.*

EXERCISE 56.

1. What is the sum of $\frac{4}{11}$ lb. Apothecaries' weight, $\frac{3}{7}$ oz. $\frac{4}{11}$ dr. and $\frac{5}{6}$ scr.? *Ans.* 4 oz. 6 drs. 2 scrs. $18\frac{172}{231}$ grs.
2. Add together $\frac{3}{5}$ yd. $\frac{1}{7}$ ell Eng. and $\frac{6}{7}$ qr. *Ans.* 3 qrs. 3 na. $1\frac{139}{140}$ in.
3. Add together $\frac{1}{7}$ of a yard, $\frac{1}{7}$ of a foot, and $\frac{1}{7}$ of an in. *Ans.* 7 inches.
4. What is the sum of $\frac{7}{11}$ of a mile, $\frac{4}{13}$ of a furlong, and $\frac{9}{22}$ of a yard? *Ans.* 5 fur. 16 rds. 0 yds. 0 ft. $3\frac{93}{143}$ in.
5. What is the sum of $\frac{1}{4}$ wk. $\frac{1}{3}$ day, $\frac{1}{5}$ h.? *Ans.* 2 days 2 h. 12 m.
6. Add together £$\frac{1}{7}$, $\frac{2}{9}$s., and $\frac{5}{12}$d. *Ans.* 3s. $1\frac{31}{84}$d.
7. What is the sum of $\frac{5}{6}$ of 21s. $\frac{1}{5}$ of 5s. $\frac{4}{5}$ of £3 12s. 6d. £$\frac{7}{13}$ and $\frac{4}{33}$d.? *Ans.* £3 12s. $4\frac{12}{13}$d.

SUBTRACTION OF VULGAR FRACTIONS.

44. Subtraction of vulgar fractions is the process of finding the difference between two fractions.

We have seen that before fractions can be added they must have a common denominator and that when denominate fractions are to be added they must be also of the same denomination, and this is manifestly the case also in the subtraction of fractions.

Hence, for the subtraction of fractions, we have the following:—

RULE.

Reduce compound and complex fractions to simple ones and all to the same denomination, if not already such.

Reduce both of the resulting fractions to a common denominator.

Subtract the numerator of the subtrahend from the numerator of the minuend, and beneath the difference write the common denominator.

NOTE.—In the case of mixed numbers it frequently happens that the fractional part of the subtrahend is greater than the fractional part of the minuend. When this occurs, instead of reducing both quantities to improper fractions and then applying the rule, it is much better to borrow *unity* from the integral part of the minuend and considering it as a fraction, having the common denominator, add it to the fractional part of the minuend. (See 3rd, 4th and 5th Examples below.)

EXAMPLE 1.—From $\frac{3}{7}$ take $\frac{2}{17}$.

$$\frac{3}{7}-\frac{2}{17}=\frac{51}{119}-\frac{14}{119}=\frac{37}{119}. \textit{ Ans.}$$

Here reducing $\frac{3}{7}$ and $\frac{2}{17}$ to a common denominator they become $\frac{51}{119}$ and $\frac{14}{119}$.

EXAMPLE 2.—From $\frac{2}{5}$ of $\frac{3}{7}$ of $\frac{10}{210}$ of 49 take $\dfrac{8\frac{2}{3}}{8\frac{2}{3}}$ of $\frac{1}{2}$ of $\frac{1}{3}$.

Here $\frac{2}{5}$ of $\frac{3}{7}$ of $\frac{10}{210}$ of $49=\frac{2}{5}$.

And $\dfrac{8\frac{2}{3}}{8\frac{2}{3}}$ of $\frac{1}{2}$ of $\frac{1}{3}=\frac{1}{6}$.

And $\frac{2}{5}-\frac{1}{6}=\frac{12}{30}-\frac{5}{30}=\frac{7}{30}$. *Ans.*

EXAMPLE 3.—From $192\frac{2}{11}$ take $16\frac{15}{16}$.

$\frac{2}{11}$ and $\frac{15}{16}$ reduced to a common denominator become $\frac{32}{176}$ and $\frac{165}{176}$.

$192\frac{2}{11}-16\frac{15}{16} = 192\frac{32}{176}-16\frac{165}{176} = 191+1\frac{32}{176}-16\frac{165}{176} = 191\frac{208}{176}-16\frac{165}{176}=175\frac{43}{176}$. *Ans.*

Here, since we cannot subtract $\frac{165}{176}$ from $\frac{32}{176}$ we have to borrow 1 from the integral part of the minuend, and considering it as $\frac{176}{176}$ add it to $\frac{32}{176}$. We thus reduce $192\frac{32}{176}$ to $191\frac{208}{176}$ and then make the subtraction.

EXAMPLE 4.—From $29\frac{2}{11}$ take $16\frac{4}{7}$.

$29\frac{2}{11} - 16\frac{4}{7} = 29\frac{14}{77} - 16\frac{44}{77} = 28 + 1\frac{14}{77} - 16\frac{44}{77} = 28\frac{91}{77} - 16\frac{44}{77} = 12\frac{47}{77}$. *Ans.*

EXAMPLE 5.—From $117\frac{3}{19}$ take $67\frac{40}{41}$.

$117\frac{3}{19} - 67\frac{40}{41} = 117\frac{123}{779}-67\frac{760}{779} = 116+1\frac{123}{779}-67\frac{760}{779} = 116\frac{902}{779}-67\frac{760}{779}=49\frac{142}{779}$. *Ans.*

EXAMPLE 6.—What is the difference between $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{5}{7}$ of $2\frac{2}{3}$ days and $\frac{4}{7}$ of $\frac{1}{3}$ of $5\frac{2}{5}$ hours?

$\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{5}{7}$ of $2\frac{2}{3}$ days $=\frac{5}{7}$ of a day $=\frac{5}{7}$ of $\frac{24}{1}$ of an hour $=\frac{120}{7}$ hours $=17\frac{1}{7}$ hours; and $\frac{4}{7}$ of $\frac{1}{3}$ of $5\frac{2}{5}$ hours $=\frac{36}{35}$ hours $=1\frac{1}{35}$ hour.

And $17\frac{1}{7}$ h. $-1\frac{1}{35}$ h. $=17\frac{5}{35}-1\frac{1}{35}=16\frac{4}{35}$ hours. *Ans.*

EXERCISE 57.

1. From $\frac{3}{4}$ take $\frac{7}{20}$. *Ans.* $\frac{2}{5}$.
2. From $\frac{205}{1496}+\frac{7}{17}$ of $\frac{3}{14}$ of $\frac{99}{11}$ take $\frac{8\frac{3}{4}}{6\frac{4}{11}}$. *Ans.* 0.
3. From $982\frac{17}{87}$ take $29\frac{19}{20}$. *Ans.* $952\frac{427}{1740}$.
4. What is the difference between $69\frac{1}{51}$ and $18\frac{56}{145}$? *Ans.* $50\frac{1603}{3773}$.
5. What is the difference between $100\frac{1}{4}$ and $9\frac{3}{8}$? *Ans.* $90\frac{7}{8}$.
6. What is the difference between $6\frac{1}{4}$ and $\frac{1}{2}$ of $9\frac{1}{4}$? *Ans.* $1\frac{5}{8}$.
7. From $611\frac{43}{191}$ take $610\frac{198}{199}$. *Ans.* $\frac{8748}{38009}$.
8. From $\frac{5}{9}$ of 2 take $\frac{4}{7}$ of $\frac{1}{5}+\frac{1}{9}$. *Ans.* $\frac{42}{45}$.
9. From $\frac{2}{3}$ of a lb. avoirdupois take $\frac{5}{6}$ of a dram.
Ans. 10 oz. $9\frac{5}{6}$ drs.
10. What is the difference between $24\frac{1}{24}$ and $21\frac{1}{21}$? *Ans.* $2\frac{167}{168}$.
11. What is the difference between $\frac{3}{8}$ of a mile, and $\frac{7}{11}$ of a furlong?
Ans. 1 fur. 5 rd. 3 yds. 1 ft. 10 in.
12. Find the value of $\frac{3}{8}$ of $\frac{125}{4}-\frac{1}{10}$ of $28\frac{1}{3}$. *Ans.* $5\frac{23}{32}$.
13. Find the value of $12\frac{3119}{17064}+\frac{1}{3}$ of $\frac{4}{7}$ of $\frac{3}{5}$ of $8\frac{1}{4}$ of $\frac{10\frac{3}{8}}{6\frac{2}{5}}-\frac{17\frac{2}{11}}{1\frac{23}{33}}$.
Ans. $2\frac{8}{9}$.
14. Find the value of $3\frac{1}{12}+8\frac{1}{9}-3\frac{3}{10}-2\frac{5}{8}+5\frac{1}{5}+6\frac{1}{9}-16\frac{1}{4}$.
Ans. $\frac{23}{48}$.
15. From $\frac{4}{11}$ of an acre take $\frac{4}{9}$ of a perch.
Ans. 1 rood 17 p. 22 yds. 2 ft. 108 in.
16. From $16\frac{1}{7}$ take $9\frac{14}{19}$, and from $169\frac{17}{100}$ take $83\frac{17}{25}$.
Ans. $6\frac{54}{133}$ and $85\frac{671}{1200}$.

MULTIPLICATION OF VULGAR FRACTIONS.

45. Let it be required to multiply $\frac{3}{11}$ by $\frac{7}{8}$.

Here we are required to multiply $\frac{3}{11}$ by $\frac{7}{8}$, that is by $\frac{1}{8}$ of 7.

Now if we multiply $\frac{3}{11}$ by 7 we shall have multiplied by a quantity 8 times too great, and the product will be 8 times too great.

If, therefore, we multiply $\frac{3}{11}$ by 7 we shall have to divide the result by 8 in order to get the product of $\frac{3}{11}\times\frac{7}{8}$.

But (Art. 8) we multiply $\frac{3}{11}$ by 7, when we multiply the numerator by 7, and we divide the result by 8 when we multiply the denominator by 8.

Therefore, $\frac{3}{11}\times\frac{7}{8}=\frac{3\times 7}{11\times 8}$, that is to multiply fractions together, we multiply the numerators together for a new numerator, and the denominators, together for a new denominator.

Hence, for the multiplication of vulgar fractions we deduce the following:—

RULE.

Reduce compound and complex fractions to simple ones (*Arts.* 31 *and* 33) *and whole and mixed numbers to improper fractions* (*Arts.* 23 *and* 25).

Cancel any factors that are common to a numerator and a denominator of the resulting fractions (*Art.* 32).

Multiply all the reduced numerators together for a new numerator, and all the reduced denominators together for a new denominator. Reduce the result, if necessary, to a mixed number.

Example 1.—Multiply $\frac{3}{8}$ by $\frac{16}{17}$.

$$\frac{3}{8}\times\frac{16}{17}=\frac{3}{1}\times\frac{2}{17}=\frac{6}{17}\ \textit{Ans.}$$

Here we cancel the first denominator and reduce the second numerator to 2.

Example 2.—Multiply together $\frac{7}{11}$, $\frac{4}{5}$, $3\frac{1}{2}$ and $\frac{55}{98}$.

STATEMENT. CANCELLED.

$$\frac{7}{11}\times\frac{4}{5}\times\frac{7}{2}\times\frac{55}{98}=\frac{\cancel{7}}{\cancel{11}}\times\frac{\overset{\cancel{2}}{\cancel{4}}}{\cancel{5}}\times\frac{\cancel{7}}{\cancel{2}}\times\frac{\overset{\cancel{5}}{\cancel{55}}}{\underset{\underset{\cancel{7}}{\cancel{49}}}{\cancel{98}}}=\frac{1}{1}\ \textit{Ans.}$$

Example 3.—Multiply together $\frac{4}{9}$, $\frac{3}{11}$, $6\frac{2}{7}$, $9\frac{3}{5}$, $2\frac{1}{2}$, and 63.

STATEMENT.

$$\frac{4}{9}\times\frac{3}{11}\times\frac{44}{7}\times\frac{48}{5}\times\frac{5}{2}\times\frac{63}{1}$$

CANCELLED.

$$\frac{\overset{2}{\cancel{4}}}{\cancel{9}}\times\frac{3}{\cancel{11}}\times\frac{\overset{4}{\cancel{44}}}{\cancel{7}}\times\frac{48}{\cancel{5}}\times\frac{\cancel{5}}{\cancel{2}}\times\frac{\overset{\cancel{7}}{\cancel{63}}}{1}=\frac{2\times3\times4\times48}{1}=1152\ \textit{Ans.}$$

Example 4.—Multiply together $\frac{1}{179}$, $18\frac{7}{11}$, $9\frac{3}{5}$, $\frac{1}{2}$ of $\frac{3}{4}$ of 7, and $\frac{3}{5}$ of $\frac{11}{14}$ of 25.

STATEMENT.

$$\frac{1}{179}\times\frac{205}{11}\times\frac{48}{5}\times\frac{21}{8}\times\frac{165}{14}$$

CANCELLED.

$$\frac{1}{179}\times\frac{205}{\cancel{11}}\times\frac{\overset{\overset{3}{\cancel{6}}}{\cancel{48}}}{\cancel{5}}\times\frac{\overset{3}{\cancel{21}}}{\cancel{8}}\times\frac{\overset{\overset{3}{\cancel{33}}}{\cancel{165}}}{\underset{\cancel{2}}{\cancel{14}}}=\frac{205\times3\times3\times3}{179}=\frac{5535}{179}=30\frac{165}{179}\ \textit{Ans.}$$

Example 5.—Multiply together $\frac{7}{9}$, $3\frac{4}{81}$, $4\frac{1}{2}$, $\frac{2}{5}$, $6\frac{1}{7}$ and $5\frac{2}{15}$.

STATEMENT.

$$\frac{7}{9}\times\frac{247}{81}\times\frac{9}{2}\times\frac{2}{5}\times\frac{43}{7}\times\frac{77}{15}.$$

CANCELLED.

$$\frac{\cancel{7}}{\cancel{9}}\times\frac{247}{81}\times\frac{\cancel{9}}{\cancel{2}}\times\frac{\cancel{2}}{5}\times\frac{43}{\cancel{7}}\times\frac{77}{15}=\frac{247\times43\times77}{81\times5\times15}=\frac{817817}{6075}=134\frac{3767}{6075}.$$

EXERCISE 58.

1. What is the product of $\frac{7}{12} \times \frac{8}{9}$? *Ans.* $\frac{9}{11}$.
2. What is the product of $\frac{3}{4} \times \frac{2}{3}$? *Ans.* $\frac{1}{2}$.
3. What is the product of $\frac{4}{15} \times \frac{5}{24}$? *Ans.* $\frac{1}{18}$.
4. Multiply together $\frac{7}{8}$, $\frac{5}{6}$ and $\frac{7}{10}$? *Ans.* $\frac{49}{96}$.
5. Multiply together 14, $15\frac{1}{16}$ and $3\frac{5}{9}$. *Ans.* $749\frac{7}{9}$.
6. Multiply together $\frac{9}{10}$, $8\frac{1}{3}$, $\frac{9}{17}$ and $\frac{11}{12}$. *Ans.* $5\frac{5}{9}$.
7. Required the product of $\frac{1}{3}$, $\frac{8}{11}$, $\frac{9}{17}$, $\frac{188}{200}$ and $\frac{5}{6}$. *Ans.* $\frac{348}{4675}$.
8. Required the product of $\frac{9}{7}$, $\frac{11}{3}$, $\frac{8}{33}$, 21, $\frac{3}{8}$ and 5. *Ans.* $18\frac{1}{3}$.
9. Required the product of $\frac{3}{8}$, $\frac{2}{3}$, $\frac{8}{17}$, $\frac{4}{19}$ and 209. *Ans.* $9\frac{1}{3}$.
10. Find the value of $6\frac{1}{4} \times 11\frac{2}{7} \times 16\frac{4}{11} \times \frac{3}{13} \times \frac{7}{50}$ of $\frac{1}{90}$. *Ans.* $\frac{3}{11}$.
11. Find the value of $\frac{4}{7}$ of $\frac{3}{11}$ of $\frac{9}{13}$ of $77 \times \frac{4}{7}$ of $\frac{3}{13}$ of $91 \times 6\frac{3}{4}$. *Ans.* $1127\frac{1}{4}$.
12. Multiply together $\frac{\frac{1}{7}}{8}$, $\frac{\frac{4}{7}}{9\frac{1}{2}}$, $\frac{7\frac{1}{3}}{8}$, $\frac{4\frac{2}{3}}{7\frac{7}{15}}$, $\frac{3}{17}$, and $1\frac{1}{3}$. *Ans.* $\frac{1}{107}$.
13. Multiply $\frac{1}{4}$ of 8 by $\frac{2}{7}$ of 19. *Ans.* $10\frac{6}{7}$.
14. Multiply $\frac{9}{10}$ of 7 by $\frac{11}{15}$ of $87\frac{3}{11}$. *Ans.* $408\frac{3}{5}$.
15. Find the value of $6\frac{1}{4} \times \frac{7}{8} \times \frac{2}{3} \times \frac{4}{7}$. *Ans.* $2\frac{7}{10}$.
16. Find the value of $3\frac{2}{3} \times 4\frac{7}{8} \times 15$. *Ans.* $268\frac{1}{8}$.
17. Multiply $\frac{1}{2}$ of $3\frac{2}{3}$ of $\frac{6}{19}$ of $9\frac{1}{2}$ by $3\frac{5}{11} \times \frac{10}{17}$ of $6\frac{1}{3}$ of $\frac{3}{8}$ of $\frac{27}{37}$ of $15\frac{1}{3}$ of $1\frac{3}{131}$. *Ans.* $4729\frac{93}{176}$.
18. Find the value of $\frac{27}{37\frac{1}{2}} \times \frac{87\frac{1}{3}}{98\frac{1}{4}} \times \frac{1\frac{3}{7}}{2\frac{1}{4}} \times \frac{81\frac{3}{11}}{123}$. *Ans.* $\frac{5}{33}$.
19. Multiply \$$38\frac{7}{11}$ by $\frac{4}{7}$ of $\frac{3}{8}$ of $\frac{11}{12}$. *Ans.* \$$8\frac{4}{7}$.
20. Find the value of $\frac{75\frac{3}{4}}{6\frac{1}{11}} \times \frac{\frac{4}{7} \text{ of } 3\frac{1}{2} \times \frac{1}{15} \text{ of } 28}{\frac{3}{11} \text{ of } 6\frac{2}{3} \times \frac{1}{17} \text{ of } 24} \times \frac{7\frac{1}{3}}{15} \times \frac{\frac{2}{3}}{\frac{4}{9}} \times 14\frac{2}{3} \times \frac{100}{121} \times \frac{4}{5\frac{1}{3}} \times \frac{\frac{4}{7}}{9}$. *Ans.* $17\frac{178}{385}$.

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46. To multiply an integral denominate number by a fraction, we have the following :—

RULE.

Multiply the denominate number by the numerator of the fraction and divide the result by the denominator.

NOTE.—This is merely considering the denominate number as a fraction having 1 for its denominator (Art. 23), and applying the preceding rule.

EXAMPLE 1.—How much is $\frac{4}{9}$ of \$129·68.

$$\tfrac{4}{9} \text{ of } \$129{\cdot}68 = \frac{\$129{\cdot}68 \times 4}{9} = \frac{\$518{\cdot}72}{9} = 57{\cdot}63\tfrac{5}{9}. \text{ Ans.}$$

EXAMPLE 2.—How much is $\frac{7}{11}$ of $\frac{1}{2}$ of 10 lb. 6 oz. 4 dr. Avoir.?

$$\tfrac{7}{11} \text{ of } \tfrac{1}{2} \text{ of 10 lb. 6 oz. 4 dr.} = \tfrac{7}{22} \text{ of 10 lb. 6 oz. 4 dr.} = \frac{\text{10 lb. 6 oz. 4 dr.} \times 7}{22} =$$

3 lbs. 4 oz. $14\frac{4}{11}$ drams. *Ans.*

EXERCISE 59.

1. How much is $1\frac{7}{36}$ of 4 days 5 h.? *Ans.* 5 days 38 m. 20 sec.
2. How much is $\frac{13}{42}$ of £29? *Ans.* £8 19s. $6\frac{6}{7}$d.
3. How much is $\frac{7}{9}$ of 186 acres 3 roods? *Ans.* 145 acres 1 rood.
4. How much is $\frac{14}{47}$ of $\frac{3}{7}$ of $\frac{1}{30}$ of $25\frac{1}{2}$ times 24 h. 30 m.? *Ans.* 1 h. 38 m.
5. How much is $\frac{4}{7}$ of $\frac{4}{9}$ of $\frac{21}{40}$ of $\frac{7}{9}$ of 33 bush. 2 pk. 1 gal.? *Ans.* 2 bush. 2 pk. 0 gal. 3 qt. $1\frac{17}{45}$ pt.

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47. From the principles already established, it is evident that—

1st. When the multiplier is less than unity, the product is less than the multiplicand.

2nd. To multiply a fraction by a whole number, we may either multiply the numerator of the fraction or divide the denominator by that number. (Art. 8.)

3rd. To multiply a whole number by any fraction having *unity* for its numerator, we simply divide the whole number by the denominator.

Thus, to multiply by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{11}$, &c., we divide by 2, 3, 4, 7, 11, &c.

4th. When multiplying by a mixed number of which the fractional part has *unity* for its numerator, it is better to multiply by the integral part of the multiplier first and then by the fractional part, afterwards adding the two partial products together.

DIVISION OF VULGAR FRACTIONS.

48. Let it be required to divide $\frac{8}{7}$ by $\frac{5}{11}$.

Here we are required to divide $\frac{8}{7}$ by $\frac{5}{11}$, that is, by $\frac{1}{11}$ of 5.

Now if we divide $\frac{8}{7}$ by 5, we use a divisor 11 times too great, and the quotient is 11 times less than the required quotient.

Therefore, to obtain the correct quotient of $\frac{8}{7}\div\frac{5}{11}$, after dividing $\frac{8}{7}$ by 5, we shall have to multiply the result by 11.

But (Art. 8) we divide the fraction $\frac{8}{7}$ by 5, when we multiply the denominator 7 by 5, and we multiply the result by 11 when we multiply the numerator 8 by 11.

Therefore $\frac{8}{7}\div\frac{5}{11}=\frac{8\times11}{7\times5}=\frac{8}{7}\times\frac{11}{5}=$ dividend × divisor with its terms inverted.

Hence for the division of fractions we have the following:—

RULE.

Reduce compound and complex fractions to simple ones; whole and mixed numbers to improper fractions.

Invert the terms of the divisor and proceed as in multiplication.

In addition to the foregoing analysis, the following may be given as a proof of the truth of this rule.

$\frac{4}{9} \div \frac{5}{11} = \frac{\frac{4}{9}}{\frac{5}{11}}$ because the dividend of any question in division may be made the numerator and the divisor the denominator of a fraction.

Now since we may multiply both terms of the fraction $\frac{\frac{4}{9}}{\frac{5}{11}}$ by *any* number, we may multiply them by $\frac{11}{5}$, i. e., the denominator with its terms inverted.

Therefore $\frac{\frac{4}{9}}{\frac{5}{11}} = \frac{\frac{4}{9} \times \frac{11}{5}}{\frac{5}{11} \times \frac{11}{5}} = \frac{\frac{4}{9} \times \frac{11}{5}}{1}$ (because $\frac{5}{11} \times \frac{11}{5} = 1$) $= \frac{4}{9} \times \frac{11}{5}$: whence the truth of the rule.

Example 1.—Divide $\frac{3}{19}$ by $\frac{4}{11}$.

$$\frac{3}{19} \div \frac{4}{11} = \frac{3}{19} \times \frac{11}{4} = \frac{33}{76} \textit{ Ans.}$$

Example 2.—Divide $\frac{3}{4}$ of $\frac{7}{11}$ by $\frac{2}{11}$ of $8\frac{3}{4}$.

$$\frac{3}{4} \text{ of } \frac{7}{11} \div \frac{2}{11} \text{ of } \frac{35}{4} = \frac{21}{44} \div \frac{35}{22} = \frac{21}{44} \times \frac{22}{35} = \frac{3}{10} \textit{ Ans.}$$

Example 3.—Divide $8\frac{4}{7}$ by $3\frac{3}{11}$.

$$8\frac{4}{7} \div 3\frac{3}{11} = \frac{60}{7} \div \frac{36}{11} = \frac{60}{7} \times \frac{11}{36} = \frac{5}{7} \times \frac{11}{3} = \frac{55}{21} = 2\frac{13}{21} \textit{ Ans.}$$

Example 4.—Divide $\frac{3}{17}$ of $\frac{4}{11}$ of $\frac{8\frac{3}{4}}{\frac{3}{11}} \times 3\frac{1}{7}$ by $\frac{4}{17}$ of $\frac{9\frac{3}{7}}{8\frac{3}{4}} \times 4\frac{3}{8}$.

STATEMENT. TERMS OF DIVISOR INVERTED.

$$\frac{3}{17} \times \frac{4}{11} \times \frac{385}{12} \times \frac{22}{7} \div \frac{4}{17} \times \frac{264}{245} \times \frac{35}{8} = \frac{3}{17} \times \frac{4}{11} \times \frac{385}{12} \times \frac{22}{7} \times \frac{17}{4} \times \frac{245}{264} \times \frac{8}{35}$$

CANCELLED.

$$= \frac{\cancel{3}}{\cancel{17}} \times \frac{\cancel{4}}{\cancel{11}} \times \frac{\overset{\cancel{11}}{\cancel{385}}}{\underset{\cancel{4}}{\cancel{12}}} \times \frac{\overset{2}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{17}}{\underset{2}{\cancel{4}}} \times \frac{\overset{35}{\cancel{245}}}{\underset{\underset{3}{\cancel{24}}}{\cancel{264}}} \times \frac{\cancel{8}}{\cancel{35}} = \frac{35}{2 \times 3} = \frac{35}{6} = \frac{5}{6} \textit{ Ans.}$$

Exercise 60.

1. Divide $\frac{1}{2}$ of $\frac{3}{5}$ by $\frac{3}{4}$ of $8\frac{3}{4}$. *Ans.* $\frac{8}{175}$.
2. Divide $\frac{15}{22}$ by $1\frac{4}{5}$ and divide the result by $\frac{5}{11}$. *Ans.* $\frac{5}{6}$.
3. Divide $82\frac{1}{17}$ by $26\frac{5}{41}$. *Ans.* $3\frac{286}{2023}$.
4. Divide $2\frac{1}{3}$ by $\frac{3}{4} + \frac{5}{8}$. *Ans.* $1\frac{2}{11}$.
5. Divide $1\frac{4}{5}$ by $\frac{1}{7}$ of $2\frac{4}{5}$ of 16 of $8\frac{3}{4}$ of $\frac{1}{70}$. *Ans.* $2\frac{1}{4}$.
6. Divide $2\frac{4}{5}$ by $(\frac{5}{9} \div \frac{5}{32}$ of 9.) *Ans.* $7\frac{7}{80}$.
7. Divide $48\frac{1}{3}$ by $\frac{2}{3} + \frac{7}{8}$ of 6. *Ans.* 19[illegible].
8. Divide $6\frac{1}{3}$ by $\frac{2}{3}$ of $\frac{9}{10} + \frac{8}{17}$. *Ans.* 6[illegible].

9. Divide $4\frac{1}{2}$ of $3\frac{1}{3}$ by $2\frac{1}{4}$ of $6\frac{1}{4}$. *Ans.* $1\frac{1}{15}$.

10. Divide $\frac{7\frac{1}{3}}{11\frac{2}{3}}$ by $\frac{\frac{3}{7}}{4\frac{1}{5}}$. *Ans.* $6\frac{17}{120}$.

11. Divide $\frac{5}{8}$ of $7\frac{3}{11}$ by $\frac{4}{11}$ of $17\frac{7}{8}$. *Ans.* $\frac{350}{549}$.

12. Divide $1\frac{17}{28}$ of $\frac{10}{13}$ of $\frac{3}{4}$ of $1\frac{2}{5}$ by $\frac{5}{6}$ of $\frac{3}{25}$ of $\frac{3}{4}$ of 5. *Ans.* $3\frac{57}{65}$.

13. Divide $\frac{1\frac{5}{6}}{4\frac{1}{3}}$ by $\frac{2\frac{1}{5}}{2\frac{1}{4}}$. *Ans.* $\frac{3}{8}$.

14. Divide $\frac{3}{25}$ by $\frac{4\frac{1}{5}}{17\frac{1}{2}}$. *Ans.* $\frac{1}{2}$.

15. Divide $14\frac{1}{5}$ of $\frac{1}{6}$ by $\frac{4}{7}$ of $8\frac{3}{13}$ of $\frac{6\frac{1}{7}}{19\frac{3}{7}}$. *Ans.* $1\frac{953}{2889}$.

16. Divide $15\frac{1}{2}$ of $\frac{\frac{4}{7}}{\frac{7}{9}}$ of $\frac{7}{\frac{2}{3}}$ of $\frac{\frac{7}{10}}{8}$ by $\frac{4\frac{5}{9}}{7}$ of $\frac{8}{4\frac{2}{5}}$ of $\frac{\frac{7}{8}}{3\frac{1}{2}}$ of $\frac{2\frac{3}{4}}{\frac{4}{7}}$.

Ans. $28\frac{473}{2205}$.

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49. To divide an integral denominate number by a fraction :—

RULE.

Multiply it by the denominator and divide the result by the numerator of the fraction.

NOTE.—This is, in effect, merely considering the denominate number as a fraction having 1 for its denominator (Art. 23) and applying the foregoing rule.

EXAMPLE.—Divide 6 days 17 hours 11 minutes by $\frac{5}{11}$.

$$6 \text{ days } 17\text{h. } 11\text{m.} \div \frac{5}{11} = 6 \text{ days } 17\text{h. } 11\text{m.} \times \frac{11}{5} = \frac{6 \text{ days } 17\text{h. } 11\text{m.} \times 11}{5}$$

$= 14$ days 18h. 36m. 12 sec. *Ans.*

EXERCISE 61.

1. Divide £8 14s. $6\frac{3}{4}$d. by $\frac{1\frac{8}{11}}{1\frac{5}{8}}$. *Ans.* £8 8s. $5\frac{1}{4}$d.

2. Divide 1m. 5 fur. 91 yds. 2 feet by $2\frac{7}{8}$ of $1\frac{9}{11}$.
Ans. 2 fur. 124 yds. 2 ft.

3. Divide 3 acres, 3 roods and 3 perches by $\frac{3}{5}$.
Ans. 6 acres 1 rood 5 per.

4. Divide £7 16s. 2d. by $\frac{4}{9}$. *Ans.* £17 11s. $4\frac{1}{2}$d.

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50. To reduce a fraction having a complex fraction in its numerator or denominator or both to a simple fraction we have simply to apply as often as necessary the rule given in Art. 33.

NOTE.—Particular attention must be paid to the relative *length* and *heaviness* of the separating lines, as they determine the various numerators and denominators.

EXAMPLE 1.—Simplify $\dfrac{\dfrac{3\frac{1}{4}}{\frac{1}{5}}}{\dfrac{7\frac{1}{2}}{\dfrac{3\frac{1}{7}}{\frac{5}{9}}}}$

OPERATION.

$$\frac{\left.\dfrac{3\frac{1}{4}}{\frac{1}{5}}\right\}}{\left.\dfrac{7\frac{1}{2}}{\left.\dfrac{3\frac{1}{7}}{\frac{5}{9}}\right\}}\right\}}=\frac{\left.\dfrac{\frac{13}{4}}{\frac{1}{5}}\right\}}{\left.\dfrac{\frac{15}{2}}{\dfrac{\frac{22}{7}}{\frac{5}{9}}}\right\}}=\frac{\frac{65}{4}}{\dfrac{\frac{15}{2}}{\frac{198}{35}}}=\frac{\frac{65}{4}}{\dfrac{15\times35}{2\times198}}=\frac{65\times2\times198}{4\times15\times35}=\frac{13\times33}{35}=12\tfrac{9}{35}$$

EXAMPLE 2.—Simplify $\dfrac{\dfrac{\frac{3\frac{1}{4}}{5}}{\frac{6}{3\frac{1}{4}}}}{\dfrac{2\frac{1}{3}}{\dfrac{8\frac{1}{2}}{\frac{5}{\frac{2}{3}}}}}$

OPERATION.

$$\frac{\left.\dfrac{\left.\frac{3\frac{1}{4}}{5}\right\}}{\left.\frac{6}{3\frac{1}{4}}\right\}}\right\}}{\left.\dfrac{2\frac{1}{3}}{\left.\dfrac{8\frac{1}{2}}{\left.\frac{5}{\frac{2}{3}}\right\}}\right\}}\right\}}=\frac{\left.\dfrac{\left.\frac{\frac{13}{4}}{\frac{5}{1}}\right\}}{\left.\frac{\frac{6}{1}}{\frac{13}{4}}\right\}}\right\}}{\left.\dfrac{\frac{7}{3}}{\left.\dfrac{\frac{17}{2}}{\left.\frac{\frac{5}{1}}{\frac{2}{3}}\right\}}\right\}}\right\}}=\frac{\left.\dfrac{\frac{13}{20}}{\frac{24}{13}}\right\}}{\left.\dfrac{\frac{7}{3}}{\left.\dfrac{\frac{17}{2}}{\frac{15}{2}}\right\}}\right\}}=\frac{\dfrac{13\times13}{20\times24}}{\dfrac{\frac{7}{3}}{\frac{17}{15}}}=\frac{\dfrac{13\times13}{20\times24}}{\dfrac{7\times5}{17}}$$

$$=\frac{13\times13\times17}{20\times24\times7\times5}=\frac{2873}{16800}\ \textit{Ans.}$$

Exercise 62.

1. Multiply $\cfrac{\cfrac{\frac{12\frac14}{7}}{3\frac14}}{\cfrac{\cfrac{9}{\frac35}}{\cfrac{\frac78}{\cfrac{5}{4\frac13}}}}$ by $\cfrac{\frac34 \text{ of } 32}{\cfrac{\cfrac{\frac78}{9\frac13}}{\cfrac{8\frac13}{\frac78}}}$ *Ans.* $2\frac{11}{117}$.

2. Divide $\cfrac{\cfrac{\frac{\frac45}{7}}{6\frac12}}{\cfrac{9\frac13}{\cfrac{3}{\frac13}}}$ by $\cfrac{\frac56}{7}$ *Ans.* $\frac{4}{135}$.

3. Divide $\cfrac{\cfrac{12\frac13}{5\frac12}}{\cfrac{3\frac34}{5\frac13}}$ by $\cfrac{\cfrac{\frac{2\frac14}{5}}{4\frac13}}{\cfrac{3\frac34}{\cfrac{16\frac23}{\frac13}}}$ *Ans.* $3\frac{14}{15}$.

51. From what has already been said, the truth of the following principles is evident.

1st. When the dividend is equal to the divisor, the quotient will be 1.

2nd. When the dividend is greater than the divisor, the quotient will be greater than 1.

3rd. When the dividend is less than the divisor, the quotient will be less than 1.

4th. The quotient will be as many times greater or less than 1 as the dividend is greater or less than the divisor.

5th. To divide a fraction by a whole number, we may either divide the numerator or multiply the denominator by that number.

6th. To divide a whole number by a fraction having 1 for its numerator, we simply multiply the whole number by the denominator of the fraction.

Thus, to divide by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c., we multiply by 2, 3, 5, 7, &c.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numerals after the Questions refer to the numbered articles of the Section.*

1. What is a fraction? (1 and 2)
2. What does every fraction indicate? (3)
3. What is the denominator of a fraction, and why is it called so? (4)
4. What is the numerator of a fraction, and why is it so called? (4)
5. What are the terms of a fraction? (5)
6. How is the value of a fraction obtained? (6)
7. When is the fraction equal to 1, and when greater or less than 1? (7)
8. What effect has multiplying the numerator of a fraction by any number? (8)
9. How does multiplying the denominator of a fraction by any number affect the value of the fraction? (8)
10. How does multiplying both terms of a fraction by the same number affect its value? (8)
11. How does dividing the numerator by any number affect the value of the fraction? (8)
12. How does dividing the denominator by any number affect the value of the fraction? (8)
13. How does dividing both numerator and denominator by the same number affect the value? (8)
14. Into what classes are fractions divided? (9)
15. What is the distinction between vulgar and decimal fractions? (10 and 11)
16. What is the meaning of the word "vulgar" as applied to fractions? (11)
17. Enumerate the six different kinds of vulgar fractions. (12)
18. What is a proper fraction? (13)
19. What is an improper fraction? (15)
20. What is a mixed number? (16)
21. To what must an improper fraction always be equal? (17)
22. What is a simple fraction? (18)
23. What is a compound fraction? (19)
24. What is a complex fraction? (21)
25. How may we convert an integer into a fraction? (23)
26. How may we reduce a whole number to a fraction having a given denominator? (24)
27. How is a mixed number reduced to an improper fraction? (25)
28. How is an improper fraction reduced to a mixed number? (26)
29. How is a fraction reduced to its lowest terms? (27 and 28)
30. How are fractions reduced to a common denominator? (29)
31. How are fractions reduced to their least common denominator? (30)
32. How is a compound fraction reduced to a simple one? (31)

33. What is meant by cancelling? (32)
34. Upon what principle may we cancel factors common to numerator and denominator? (32 and 8)
35. How do we reduce complex fractions to simple ones? (33)
36. What is a denominate fraction? (34)
37. In what does reduction of denominate fractions consist? (35)
38. How do we reduce a denominate fraction from a lower to a higher denomination? (36)
39. How do we reduce a denominate fraction from a higher to a lower denomination? (37)
40. How do we find the value of a denominate fraction? (38)
41. How do we reduce one denominate number to the fraction of another? (39)
42. What is addition of fractions? (40)
43. What kind of fractions only can be added? (41)
44. What is the rule for addition of fractions? (42)
45. When mixed numbers are to be added how do we proceed? (42, note)
46. What is the rule for the addition of denominate fractions? (43)
47. What is the rule for the subtraction of fractions? (44)
48. What is the rule for multiplication of fractions? (45)
49. Give a proof of the truth of this rule. (45)
50. How do we multiply an integral denominate number by a fraction? (46)
51. How may we multiply a fraction by a whole number? (47)
52. How do we multiply a whole number by a fraction having 1 for numerator? (47)
53. How do we multiply a whole number by a mixed number, the fractional part of which has 1 for numerator? (47)
54. What is the rule for division of fractions? (48)
55. Give a proof of the truth of this rule. (48)
56. How do we divide an integral denominate number by a fraction? (49)
57. How do we divide a fraction by a whole number? (51)
58. How do we divide a whole number by a fraction having 1 for its numerator? (51)

EXERCISE 63.

MISCELLANEOUS EXERCISE ON VULGAR FRACTIONS.

1. The Ottawa River is 800 miles long; the Gatineau 420 miles, the Chaudière 100 miles, the Richelieu 160 miles, and the Niagara 35 miles. The entire length of the St. Lawrence, from the upper end of Lake Superior to the Sea is 2000 miles. How will the lengths of these different rivers be expressed as fractions of that of the St. Lawrence?
2. The population of Goderich is $\frac{3}{8}$ of that of Peterborough, the population of Peterborough is $1\frac{1}{4}$ of that of Brockville, the population of Brockville is $1\frac{3}{8}$ of that of Prescott, the population of Prescott is $\frac{1}{4}$ of that of Ottawa City, the population of Ottawa City is $2\frac{1}{3}$ of that of Port Hope, and the population of Port Hope is $\frac{4}{45}$ of that of Toronto. What fraction is the population of Goderich of that of Toronto?
3. What will $6\frac{1}{3}$ pounds of tea cost, at $65\frac{3}{4}$ cents per lb.?
4. Suppose I have $\frac{3}{8}$ of a ship, and that I buy $\frac{4}{11}$ more; what is my entire share?

5. A boy divided his marbles in the following manner: he gave to A $\frac{1}{3}$ of them, to B $\frac{1}{10}$, to C $\frac{1}{5}$, and to D $\frac{1}{6}$, keeping the rest to himself; how many did he give away, and how many did he keep?

6. Find the value of $\dfrac{5\frac{3}{8}-2\frac{1}{8}}{3\frac{3}{4}+\frac{9}{20}}$ of $\dfrac{4\frac{1}{2}+5\frac{19}{25}}{4\frac{1}{20}}$ of $\dfrac{2\frac{2}{3}+1\frac{3}{8}}{7\frac{19}{24}-2\frac{1}{4}}$

7. What cost 1670 $\frac{7}{15}$ pounds of coffee at $12\frac{3}{4}$ cents per pound?

8. A tree whose length was 136 feet, was broken into two pieces by falling; $\frac{2}{3}$ of the length of the longer piece equalled $\frac{3}{4}$ of the length of the shorter. What was the length of the two pieces respectively?

9. A farmer bought at one time $97\frac{1}{4}$ acres of land, for 1000 dollars; at another, $127\frac{2}{5}$ acres for $1375\frac{1}{2}$ dollars; at another, $500\frac{3}{8}$ acres for 6831 dollars; and at another, $383\frac{1}{3}$ acres for $4013\frac{3}{16}$ dollars. What was the whole quantity of land that he purchased, and the sum that he paid for it?

10. Find the value of $(12\frac{5}{6}-8\frac{3}{4}-1\frac{1}{10}+\frac{8}{15})\times 4\frac{1}{2}\times(7\frac{5}{12}-6\frac{1}{2})$, and also of $(\frac{2}{3}+1\frac{4}{7})-(\frac{4}{5}+3\frac{3}{11})$.

11. What is the value of $19\frac{7}{8}$ barrels of flour at \$$6\frac{3}{4}$ a barrel?

12. What is the value of $876\frac{11}{16}$ acres of land, at \$$75\frac{3}{8}$ per acre?

13. Bought at one time $147\frac{2}{3}$ bushels of coal, and at another time $320\frac{1}{5}$ bushels. Having consumed $156\frac{1}{4}$ bushels, I desire to know what quantity of the coal purchased is still on hand.

14. Divide $\dfrac{7\,(1\frac{1}{2}\text{ of }\frac{3}{4})}{\frac{1}{8}\left(\dfrac{3}{3\frac{1}{3}}\text{ of }7\right)}$ by $7\frac{1}{8}$; and find the value of $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\dfrac{1}{2\frac{1}{3}}+\dfrac{1}{3\frac{1}{2}}+\dfrac{1}{4\frac{1}{3}}}$

15. If $17\frac{4}{5}$ bushels of wheat sow $7\frac{4}{5}$ acres, how many bushels will it require to sow one acre?

16. Multiply the sum of $3\frac{5}{8}$, $4\frac{3}{4}$, and $4\frac{3}{8}$, by the difference of $7\frac{4}{5}$ and $5\frac{7}{8}$, and divide the product by the sum of $94\frac{1}{8}$ and $93\frac{1}{6}$.

17. Divide 2 by the sum of $2\frac{3}{5}$, $\frac{1}{3}$, and 4; add $1\frac{2}{3}-\frac{7}{8}$ to the quotient; and multiply the result by the difference of $5\frac{1}{3}$ and $4\frac{1}{2}$.

18. Find the value of $(\frac{1}{2}+\frac{1}{3})\times(1\frac{1}{3}+2\frac{3}{4})\times(2\frac{1}{12}-1\frac{1}{8})\times(3\frac{1}{10}-\frac{4}{5})$; and also of $(1\frac{3}{4}+2\frac{1}{2})\div(5\frac{1}{3}+3\frac{1}{8})$.

19. A person dies worth \$40000, and leaves $\frac{1}{3}$ of his property to his wife, $\frac{1}{5}$ to his son, and the rest to his daughter. The wife at her death leaves $\frac{2}{5}$ of her legacy to the son, and the rest to her daughter; but the son adds his fortune to his sister's and gives her $\frac{1}{3}$ of the whole. How much will the sister gain by this, and what fraction will her gain be of the whole?

DECIMALS AND DECIMAL FRACTIONS.

52. A decimal fraction is a fraction having unity with one or more 0s to the right of it for denominator:

Thus $\frac{4}{1000}$, $\frac{7}{100}$, $\frac{8}{10}$, $\frac{17}{1000000}$, &c., are decimal fractions.

53. A decimal fraction is reduced to its corresponding *decimal* by dividing the numerator by the denominator; but since (Art. 52) this denominator is unity followed by one or more 0s, we divide the numerator by the denominator when we move the decimal point as many places to the left in the numerator as there are 0s in the denominator.

Example 1. Reduce $\frac{743}{1000}$ to a decimal. *Ans.* ·743.

2. Reduce $\frac{92376}{100000000}$ to a decimal. *Ans.* ·00092376.

Exercise 64.

1. Reduce $\frac{567}{1000}$, $\frac{98}{100000}$ and $\frac{7}{10}$ to decimals. *Ans.* ·567, ·00098 and ·7.
2. Reduce $\frac{23}{10000000}$ and $\frac{176}{10000000}$ to decimals. *Ans.* ·0000023 and ·0000176.
3. Reduce $\frac{278643}{1000000000}$ to a decimal. *Ans.* ·000278643.

---

54. It is as inaccurate to confound a decimal fraction with its corresponding decimal as to confound a vulgar fraction with its quotient: Thus the value of $\frac{3}{4}$ is ·75, so also the value of $\frac{75}{100}$ is ·75, but ·75 and $\frac{75}{100}$ are no more *identical* than are $\frac{3}{4}$ and ·75.

55. To reduce a decimal to its corresponding decimal fraction:—

RULE.

Consider the significant part of the decimal as numerator and beneath it write for denominator 1 *followed by as many 0s as there are places in the decimal.*

Example 1. Reduce ·043 to a decimal fraction. *Ans.* $\frac{43}{1000}$.

2. Reduce ·00000576 to a decimal fraction. *Ans.* $\frac{576}{100000000}$.

Exercise 65.

1. Reduce ·73, ·092 and ·0003 to decimal fractions. *Ans.* $\frac{73}{100}$, $\frac{92}{1000}$, and $\frac{3}{10000}$.
2. Reduce ·137 and ·000006943 to decimal fractions. *Ans.* $\frac{137}{1000}$, and $\frac{6943}{1000000000}$.
3. Reduce ·13578967 and ·023004008 to decimal fractions. *Ans.* $\frac{13578967}{100000000}$, and $\frac{23004008}{1000000000}$.

56. Decimal fractions follow exactly the same rules as vulgar fractions. It is, however, generally more convenient to obtain their quotients, and then perform on them the required processes of addition, &c., by the methods already described (Sect. II).

To reduce a vulgar fraction to a decimal or to a decimal fraction—

RULE.

Divide the numerator by the denominator and the quotient will be the required "decimal"; the latter may be changed to its corresponding decimal fraction by (Art. 55).

This is merely actually performing the division which the fraction indicates.

EXAMPLE 1.—Reduce $\frac{7}{8}$ to a decimal and also to a decimal fraction.

$$\begin{array}{r} 8)\underline{7\cdot} \\ \cdot875 \end{array} \textit{ Ans.} = \tfrac{875}{1000} \textit{ Ans.}$$

2. Reduce $\frac{9}{16}$ to a decimal.

$$\begin{array}{r} 16)\underline{9\cdot} \\ \cdot5625 \end{array} \textit{ Ans.}$$

EXERCISE 66.

1. Reduce $\frac{1}{2}$ and $\frac{3}{8}$ to decimals. *Ans.* ·5 and ·375.
2. Reduce $\frac{9}{25}$ and $\frac{1}{4}$ to decimal fractions. *Ans.* $\frac{36}{100}$ and $\frac{25}{100}$.
3. Reduce $\frac{73}{75}$, $\frac{574}{123}$, and $\frac{15}{34}$ to decimals. *Ans.* ·9733+, 4·666+and ·44117+.*
4. Reduce $\frac{6}{7}$, $\frac{5}{12}$, and $\frac{4}{9}$ to decimals. *Ans.* ·857142+, ·4166 + and ·44444+.
5. Reduce $\frac{17}{112}$ and $\frac{718}{1296}$ to decimals. *Ans.* ·15178571428+and ·554012+.

---

57. Let it be required to reduce £3 7s. 6¾d. to the decimal of a pound.

OPERATION.

¾d=·75d hence 6¾d=6·75d. If now we divide this by 12 we shall have its value as the decimal of a shilling.
6¾d=6·75d=·5625s. hence 7s 6¾d=7·5625s.
Next if we divide this by 20 we shall have its value as a decimal of a pound.
7s. 6¾d=7·5625s=£·378125.
Therefore £3 7s 6¾d=£3·378125.

Hence to reduce a denominate number of different denominations to an equivalent decimal of a given denomination we deduce the following—

---

* The sign + written after these answers simply indicates that there is still a remainder and consequently that the division may be carried on further.

RULE.

Divide the lowest denomination named by that number which makes one of the next higher denomination.

Annex this quotient to the number of the next higher denomination given and divide as before.

Proceed thus through all the denominations to the one required, and the last result will be the one sought.

Example 1.—Reduce 3 days, 12 hours, 3 minutes, 30 seconds to the decimal of a week.

60)30=sec.=30 sec.
60)3·5=decimal of a minute=3 min. 30 sec.
24)12·0583=decimal of an hour=12 h. 3 m. 30 sec.
7)3·5024305=decimal of a day=3 days 12 h. 3 m. 30 sec.
Ans. ·5003472=decimal of a week=3 days 12 h. 3 m. 30 sec.

Example 2.—Reduce 187 lb. 13 oz. 11 drams to the decimal of a ton.

OPERATION.

16)11· drams.
16)13·6875 ounces.
2000)187·85546875 lbs.
·093927734375 ton. *Ans.*

Here we divide the 11 drams by 16 and thus obtain ·6875 to which we prefix the given 13 oz. Next we divide this by 16 and obtain ·85546875 to which we bring down the 187 lb. and divide the result by 2000, the number of lbs. in a ton.

Note.—To divide by 2000 remove the decimal point *three* places to the left and divide by 2; similarly to divide by 60, 20, &c., remove the decimal point *one* place to the left and divide by 6, 2, &c.

Exercise 67.

1. Reduce 3 yds. 2 ft. 1 in. to the decimal of a furlong. *Ans.* ·01679+.
2. Reduce 3 dwt. 17 grs. Troy, to the decimal of a pound. *Ans.* ·01545138+.
3. Reduce 2 scr. 7 grs. to the decimal of a pound, Apoth. *Ans.* ·0081597+.
4. Reduce 5 fur. 35 per. 2 yd. 2 ft. 9 in. to the decimal of a mile. *Ans.* ·73603+.
5. Reduce 3 qr. 2 na. to the decimal of a yard. *Ans.* ·875.
6. Reduce 5s. to the decimal of 13s. 4d. *Ans.* ·375.*

---

* Reduce 5s. first to the fraction of 13s. 4d. and then reduce the resulting fraction to a decimal.

Thus 5s. reduced to the fraction of 13s. 4d.=$\frac{60}{160}$=$\frac{3}{8}$=·375.

7. Reduce 12 h. [illegible] 11 sec. to the decimal of a day.
Ans. ·5384875.

8. Reduce [illegible] of [illegible] of [illegible] d. to the decimal of £[illegible]. *Ans.* ·012053+.

9. Reduce [illegible] of [illegible] of a mile to the decimal of 3½ inches.
Ans. 3620·571428+.

10. Reduce [illegible] of [illegible] of 8¼ lb. Avoir. to the decimal of [illegible] of an oz.
Ans. 9·2444+.

11. Reduce 3 pk. 1 gal. 1 qt. 1 pt. to the decimal of a bushel.
Ans. ·921875.

---

58. Let it be required to find the value in terms of a lower denomination of ·7825 of a yard.

OPERATION.

·7825
3

2·3475
12

4·1700
12

2·0400
Ans. 2 ft. 4 in. 2·04 lines.

EXPLANATION.—Since there are 3 feet in a yard, it is evident that any decimal of a yard is *three* times as great a decimal of a foot. Hence to reduce the decimal of a yard to a decimal of a foot we multiply it by 3. This gives us two feet and ·3475 of a foot. Similarly multiplying the decimal of a foot by 12 reduces it to an equivalent decimal of an inch. We thus find ·3475 of a foot equal to 4 inches and ·17 of an inch. Again, multiplying this last by 12 reduces it to the decimal of a line, and we thus find the whole quantity ·7825 of a yard equal to 2 ft. 4 in. 2·04 lines.

NOTE.—In these multiplications we only multiply the number to the right of the separating point.

Hence, to find the value of a denominate number in terms of integers of a lower denomination we have the following—

RULE.

Multiply the given decimal by the number of units of the next lower denomination that make one of the given denomination.

Point off as many decimal places as there were in the multiplier, and the integral portion, if any, will be units of that lower denomination; the decimal part may be reduced to a still lower denomination, and so on.

EXAMPLE 1.—Find the value of £·97875.

OPERATION.

·97875
20

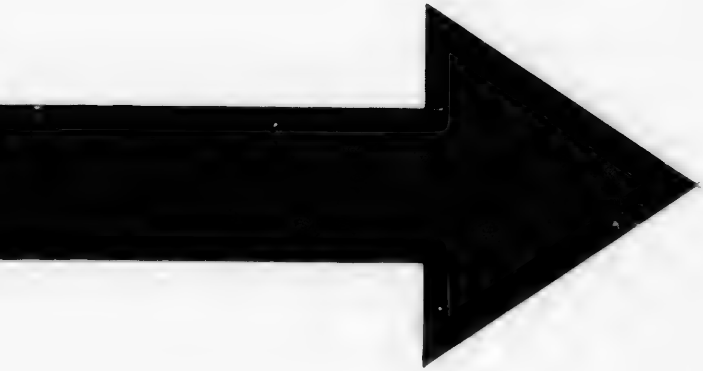
19·57500s.
12

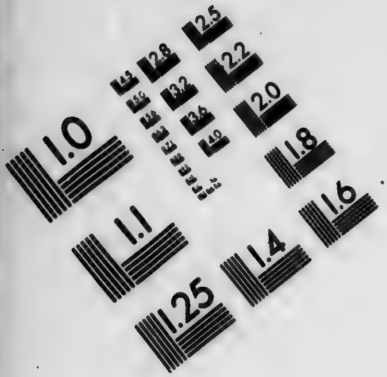
6·90000d.
4

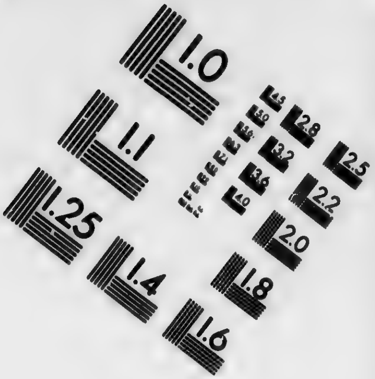
[illegible]

Ans. 19s. 6¾d. + [illegible] of a farthing.

[redacted]







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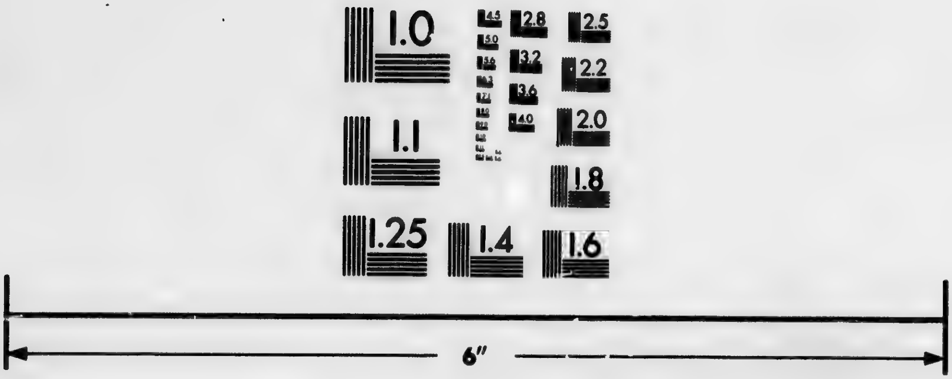

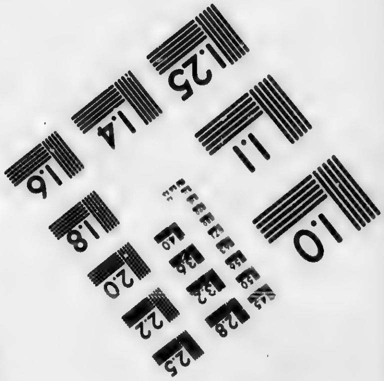

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4.5
2.8
5.0
5.6
3.2
2.5
3.6
2.2
4.0
2.0
1.8
1.1
1.0

Example 2.—Find the value of ·7863625 of a pound Apothecaries weight.

OPERATION.
·7863625
12
9·4363500 oz.
8
3·4908000 drs.
3
1·4724000 scr.
20
9·4480000 grs.

Ans. 9 oz. 3 dr. 1 scr. 9·448 grains.

Exercise 68.

1. Find the value of 0·3945 of a day. *Ans.* 9 hours 28 min. 4·8 sec.
2. Find the value of 0·3965 of a mile. *Ans.* 3 fur. 6 per. 4 yds. 2 ft. 6.24 in.
3. Find the value of 0·309153 of an oz. Troy. *Ans.* 6 dwt. 4·39344 grains.
4. Find the value of 22·75 of £2 2s. 6d. *Ans.* £48 6s. 10½d.
5. Find the value of 11·17825 of 7 bush. 1 pk. 1 gal. 1 qt. *Ans.* 82 bush. 3 pks. 0 gal. 1 qt. 0·4905 pt.*
6. Find the value of ·2057 of a lb. Troy. *Ans.* 2 oz. 9 dwt. 8·832 grains.
7. Find the value of ·176 of 1 fur. 36 per. 2 yds. 5 in. *Ans.* 13 per. 2 yds. 1 ft. [illegible] in.
8. Find the value of ·625 of a league. *Ans.* 1 mile 7 fur.
9. What is the value of ·015625 of a bushel? *Ans.* 1 pint.
10. What is the value of ·9378 of an acre? *Ans.* 3 roods 30 per. 1 yd. 4 ft. 9$\frac{99}{125}$ inches.
11. Find the value of ·2775 of 1 sq. yd. 3 ft. 72 in. *Ans.* 3 sq. ft. 67½ in.

CIRCULATING OR REPEATING DECIMALS.

59. Let it be required to reduce $\frac{5}{9}$ and $\frac{6}{7}$ to decimals.

OPERATION.
9)5
·555555, &c.
7)6
·857142857142857142, &c.

---

* If the given quantity be expressed in more than one denomination it should be reduced to *one* before applying the rule. Thus in this example 7 bush. 1 pk. 1 gal. 1 qt.=237 qts. and 11·17825 × 237=2649·24525 qts.=82 bush. 3 pks. 0 gal. 1 qt. 0·4905 pints.

In these and many other cases the division does not terminate, and the value of the fraction can only be approximately expressed. In the former of the above examples the figure 5 is constantly repeated, and in the latter the series of figures 857142.

60. Decimals which do not terminate, *i. e.*, which consist of the same digit or set of digits constantly repeated, are called Repeating or Circulating Decimals.

61. The digit or set of digits, which repeats, is called a *repetend*, *period* or *circle*.

NOTE.—The terms *period* and *circle* are commonly used only when the repetend contains two or more digits.

62. A single Repetend is one in which only a single digit repeats,

Thus ·3333 &c.; ·7777 &c.; ·33333 &c. are single repetends.

63. A single Repetend is expressed by writing the digit that repeats with a dot over it,

Thus, ·333 &c. is written $\cdot\dot{3}$; ·777 &c. is written $\cdot\dot{7}$.

64. A Circulating Decimal or Compound Repetend is one in which more than one digit repeats,

Thus, ·347347347 &c.; ·202020 &c.; ·123412341234, &c., are Circulating Decimals or Compound Repetends.

65. A Circulating Decimal is expressed by writing the recurring period once with a dot over its first and last digits,

Thus, ·347347 &c. is written $\cdot\dot{3}4\dot{7}$; ·2020 &c. $\cdot\dot{2}\dot{0}$; ·12341234 &c. is written $\cdot\dot{1}23\dot{4}$.

66. A Pure Repetend or Circulating Decimal is one in which the repetend commences *immediately* after the decimal point.

67. A Mixed Repetend or Circulating Decimal is one which contains one or more ciphers or significant figures between the repetend and the decimal point,

Thus, $\cdot\dot{3}$, $\cdot\dot{7}$, $\cdot\dot{1}$ are Pure Repetends.

$\cdot 7891\dot{7}$, [illegible], [illegible] are Mixed Repetends.

$\cdot\dot{1}\dot{2}$, $\cdot\dot{0}4\dot{3}$, $\cdot\dot{3}167\dot{3}$ are Pure Circulating Decimals.

·1872, ·67803, ·0717836 are Mixed Circulating Decimals.

68. Similar Repetends are those which commence at the same number of places from the decimal point,

Thus, $\cdot 718\dot{4}\dot{5}$, $\cdot 912\dot{7}8\dot{6}$ and $\cdot 000\dot{7}184\dot{6}$ are Similar Repetends.

69. Dissimilar Repetends are those which commence at a different number of places from the decimal point,

Thus, $\cdot 7\dot{8}4\dot{2}$, $\cdot 928\dot{6}2\dot{7}$ and $\cdot 9184\dot{2}7\dot{8}$ are Dissimilar Repetends.

70. Coterminous Repetends are those which terminate at the same number of places from the decimal point,

Thus, $\cdot\dot{7}43\dot{7}$, $\cdot 62\dot{4}\dot{3}$ and $\cdot\dot{1}84\dot{7}$ are Coterminous Repetends.

71. Similar and Coterminous Repetends are those which both commence and end at the same distance from the decimal point,

Thus, $\cdot 78\dot{4}26\dot{7}$, $16\cdot 47\dot{1}21\dot{2}$, $198\cdot 16\dot{1}84\dot{1}$ are Similar and Coterminous Repetends.

72. In reducing a fraction to a decimal we place a point after the numerator, and annex 0s to it until it is exactly divisible by the denominator. But since the point does not affect the division, merely determining the *place* of the point in the resulting quotient, it is manifest that we may leave it altogether out of consideration, so that annexing 0s to the numerator becomes in effect *multiplying it by such a power of* 10 *as will make it contain the denominator.* Now if the fraction, before proceeding to the division, be reduced to its lowest terms, the denominator can have no factor in common with the numerator; and if the denominator be exactly contained in the numerator with the 0s annexed, it can only be from its being contained in that power of 10 by which the original numerator was multiplied. But since 10 contains only the factors 2 and 5, any power of 10 can contain only the factors 2 and 5; and hence, in order that the denominator may be exactly contained in the numerator with 0s annexed, it must contain only the factors 2 and 5, or powers of 2 and 5.

Hence, when a vulgar fraction is reduced to its lowest terms, if the denominator contain no factors other than 2 and 5, the corresponding decimal will be *finite*; but if the denominator contain any other factor than 2 and 5, as 3, 7, 11, &c., the corresponding decimal will be *infinite*, i. e., will be a repetend.

Example.—Can $\frac{7}{16}$, $\frac{11}{25}$, $\frac{5}{12}$ and $\frac{17}{125}$ be exactly expressed as decimals?

16, the denominator of the first, $= 2 \times 2 \times 2 \times 2$, (i. e. contains no prime factor other than 2 or 5) therefore it *can* be exactly expressed by a decimal.

$25 = 5 \times 5$ (i. e. no prime factor other than 2 or 5) therefore $\frac{11}{25}$ *can* be exactly expressed by a decimal.

$12 = 2 \times 2 \times 3$ (i. e. *does* contain a factor other than 2 or 5) therefore $\frac{5}{12}$ *cannot* be exactly decimated.

$125 = 5 \times 5 \times 5$ (i. e. no factor other than 2 or 5) therefore $\frac{17}{125}$ *can* be exactly decimated.

EXERCISE 69.

Of the following fractions, which can and which cannot be exactly decimated, i. e., reduced to equivalent decimals?

1. $\frac{1}{4}$, $\frac{17}{625}$, $\frac{11}{32}$, $\frac{31}{1024}$, and $\frac{173}{800}$.
2. $\frac{6}{1175}$, $\frac{1}{4}$, $\frac{7}{25}$, $\frac{8}{500}$, $\frac{111}{264}$.
3. $\frac{11}{24}$, $\frac{6}{11}$, $\frac{7}{15}$, $\frac{3}{8}$, and $\frac{117}{1250}$.

73. We may determine the number of places in the decimal or finite part of the decimal corresponding to a vulgar fraction by the following—

RULE.

Reduce the fraction to its lowest terms, and decompose the denominator into its prime factors.

If the denominator contains no factors other than 2 or 5, or powers of 2 or 5, the whole decimal is finite.

If the denominator does not contain 2 or 5 as factor, the decimal contains no finite part.

The highest exponent of 2 or 5 will indicate the number of decimal places in the finite part of the corresponding decimal.

EXAMPLE 1.—How many decimal places will be required to express $\frac{427}{3125}$?

Here, $3125=5\times5\times5\times5\times5=5^5$. Therefore the equivalent decimal will contain 5 places.

EXAMPLE 2.—How many decimal places will be required to express $\frac{343}{1600}$?

Here, $1600=2\times2\times2\times2\times2\times2\times5\times5=2^6\times5^2$. Hence 6 is the highest exponent, and the number of decimal places will therefore be 6.

EXERCISE 70.

1. How many decimal places will be required to express the following fractions, viz:—$\frac{11}{16}$, $\frac{9}{40}$, $\frac{111}{8000}$ and $\frac{133}{1024}$?

   Ans. 4, 3, 6 and 10.

2. How many places will there be in the finite part of the decimals corresponding to $\frac{7}{96}$, $\frac{111}{896}$, $\frac{427}{15120}$ and $\frac{133}{6144}$?

   Ans. 5, 7, 4 and 11.

---

74. In decimating vulgar fractions where many places are required in the decimal, the method of continually dividing becomes very tedious. In such cases we may sometimes shorten the work as follows:—

EXAMPLE.—What decimal is equivalent to the vulgar fraction $\frac{1}{17}$?

OPERATION.

$$\begin{array}{l} 29)1\cdot00(0\cdot03448 \\ 87 \\ \hline 130 \\ 116 \\ \hline 140 \\ 116 \\ \hline 240 \\ 232 \\ \hline 8 \end{array}$$

$\frac{1}{29}=0\cdot03448\frac{8}{29}$. Therefore $\frac{8}{29}=0\cdot27586\frac{6}{29}$ and substituting this value for $\frac{8}{29}$ we get:—

$\frac{1}{29}=0\cdot0344827586\frac{6}{29}$. Hence $\frac{6}{29}=0\cdot2068965517\frac{7}{29}$ and substituting this for $\frac{6}{29}$ we get:—

$\frac{1}{29}=0\cdot03448275862068965517\frac{7}{29}$. Hence $\frac{7}{29}=0\cdot241379310344827\text{-}58620\frac{23}{29}$ and substituting this value for $\frac{7}{29}$ we get:—

$\frac{1}{29}=0\cdot\dot{0}34482758620689655172413793\dot{1}$. *Ans.*

75. The number of places in a period cannot exceed the units in the denominator minus one.

This is manifest from the fact that all the remainders that occur must be less than the denominator, and their number cannot be greater than the denominator, minus one; because we carry on the division by affixing 0s, and it follows that whenever we obtain a remainder like one that has previously occurred, the digits of the decimal will begin to repeat.

Thus $\frac{6}{7}=0\cdot\overset{4}{8}\overset{5}{5}\overset{1}{7}\overset{3}{1}\overset{2}{4}\overset{6}{2}$, where the small figures above the line represent the successive remainders, none of which, of course, can be as great as 7, the divisor,—the next remainder after the 6 would be 4, and consequently the digits would commence to repeat.

76. Those repetends that have as many places, minus one, as there are units in the denominators of their equivalent vulgar fractions are sometimes called *perfect repetends.*

The following are the only fractions having a denominator less than 100 that give *perfect repetends* when decimated:—

$\frac{1}{7}$, $\frac{1}{17}$, $\frac{1}{19}$, $\frac{1}{23}$, $\frac{1}{29}$, $\frac{1}{47}$, $\frac{1}{59}$, $\frac{1}{61}$ and $\frac{1}{97}$.

77. To reduce a pure repetend to an equivalent vulgar fraction:—

RULE.

Put the period for numerator, and as many nines as there are places in the period for denominator.

Example.—What vulgar fractions are equivalent to $\cdot\dot{7}$, $\cdot\dot{9}\dot{3}$, $\cdot\dot{7}0\dot{4}$ and $\cdot\dot{0}0704\dot{3}$?

Ans. $\cdot\dot{7}=\frac{7}{9}$; $\cdot\dot{9}\dot{3}=\frac{93}{99}=\frac{31}{33}$; $\cdot\dot{7}0\dot{4}=\frac{704}{999}$; $\cdot\dot{0}0704\dot{3}=\frac{7043}{999999}$.

Reason $\frac{1}{9}=\cdot\dot{1}$ therefore $\frac{2}{9}, \frac{3}{9}, \frac{4}{9}$, &c., $=\cdot\dot{2}, \cdot\dot{3}, \cdot\dot{4}$, &c., hence $\cdot\dot{1}, \cdot\dot{2}, \cdot\dot{3}$, &c., $=\frac{1}{9}, \frac{2}{9}, \frac{3}{9}$.

Similarly $\frac{1}{99}=\cdot\dot{0}\dot{1}$, therefore $\frac{7}{99}=\cdot\dot{0}\dot{7}$; $\frac{23}{99}=\cdot\dot{2}\dot{3}$; $\frac{79}{99}=\cdot\dot{7}\dot{9}$; &c.

Hence $\cdot\dot{0}\dot{1}=\frac{1}{99}$; $\cdot\dot{0}\dot{7}=\frac{7}{99}$; $\cdot\dot{2}\dot{3}=\frac{23}{99}$; $\cdot\dot{1}\dot{7}=\frac{17}{99}$; &c.

So also $\frac{1}{999}=\cdot\dot{0}0\dot{1}$; $\frac{5}{999}=\cdot\dot{0}0\dot{5}$; $\frac{167}{999}=\cdot\dot{1}6\dot{7}$; &c.

Hence $\cdot\dot{0}0\dot{1}=\frac{1}{999}$; $\cdot\dot{2}4\dot{3}=\frac{243}{999}$; &c., whence the reason of the rule is evident.

Exercise 71.

1. Reduce $\cdot\dot{8}$, $\cdot\dot{0}\dot{5}$, $\cdot\dot{3}4\dot{2}$, $\cdot\dot{7}00\dot{4}$ and $\cdot\dot{0}0200\dot{3}$ to equivalent vulgar fractions.
   Ans. $\frac{8}{9}$, $\frac{5}{99}$, $\frac{342}{999}=\frac{38}{111}$, $\frac{7004}{9999}$ and $\frac{2003}{999999}$.
2. Reduce $\cdot\dot{1}\dot{9}$, $\cdot\dot{1}06\dot{7}$, $\cdot\dot{1}111\dot{5}$ and $\cdot\dot{7}0410\dot{3}$ to equivalent vulgar fractions.
   Ans. $\frac{19}{99}$, $\frac{1067}{9999}=\frac{97}{909}$, $\frac{11115}{99999}=\frac{1235}{11111}$ and $\frac{704103}{999999}=\frac{234701}{333333}$.
3. Reduce $\cdot\dot{1}0\dot{2}$, $\cdot\dot{0}01\dot{8}$, $\cdot\dot{0}000710\dot{3}$, $\cdot\dot{0}102030\dot{4}$ and $\cdot\dot{9}8765432\dot{1}$ to equivalent vulgar fractions.
   Ans. $\frac{34}{333}$, $\frac{18}{9999}$, $\frac{7103}{99999999}$, $\frac{1020304}{99999999}$ and $\frac{109739369}{111111111}$.

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78. To reduce a mixed repetend to an equivalent vulgar fraction—

RULE.

Subtract the finite part from the whole and set down the difference for the numerator.

For denominator put as many 9s as there are places in the 'infinite' part followed by as many 0s as there are places in the 'finite' part.

Example.—Reduce $\cdot7\dot{3}$, $\cdot12\dot{3}\dot{4}$ and $\cdot713\dot{2}09\dot{2}$ to their equivalent vulgar fractions.

OPERATION.

73 − 7 = 66 = numerator of *first* fraction.
1234 − 12 = 1222 = " *second* "
7132092 − 713 = 7131379 = " *third* "
90 = 1st Denominator, since the repetend contains one place in the finite, and one place in the infinite part.
9900 = 2nd Denominator, since the repetend contains two places in the finite part and two in the infinite part.
9999000 = 3rd Denominator, since the infinite part of the decimal contains *four* places and the finite part *three* places,

Hence, $\cdot7\dot{3}=\frac{66}{90}=\frac{11}{15}$, $\cdot1\dot{2}3\dot{4}=\frac{1222}{9900}=\frac{611}{4950}$ and $\cdot713\dot{2}09\dot{2}=\frac{7131379}{9999000}$.

Reason.—Let it be required to reduce $\cdot97\dot{8}73\dot{4}$ to an equivalent vulgar fraction.

$$\text{Let } x=\cdot97\dot{8}73\dot{4} \qquad \text{(I)}$$

$$\text{Then } 100\,x=97\cdot\dot{8}73\dot{4} \qquad \text{(II)}$$

And $1000000\,x=978734\cdot\dot{8}73\dot{4}$ (III); subtracting (II) from (III) gives
$999900\,x=978734-97$.

Whence $x=\dfrac{978734-97}{999900}$ = *Whole repetend minus the finite part for a numerator; and as many 9s as there are places in infinite part, followed by as many 0s as there are places in finite part for denominator.*

The rule may also be explained as follows:—

Taking the same example $\cdot97\dot{8}73\dot{4}$ and multiplying it by 100, we get $\cdot97\dot{8}73\dot{4}\times100=97\cdot\dot{8}73\dot{4}=97+\cdot\dot{8}73\dot{4}=97+\frac{8734}{9999}$ (Art. 77.)

Now, since we multiplied by 100 this result is 100 times too great. Therefore $\cdot97\dot{8}73\dot{4}=\frac{97}{100}+\frac{8734}{999900}$ and to add these fractions we must reduce them to a common denominator when they become:

$$\frac{97\times9999}{999900}+\frac{8734}{999900}=(\text{since } 9999=10000-1)$$

$$\frac{97\times(10000-1)}{999900}+\frac{8734}{999900}=\frac{97\times10000-97}{999900}+\frac{8734}{999900}=\frac{970000-97}{999900}+\frac{8734}{999900}$$

$=\dfrac{978734-97}{999900}$ = *Whole repetend minus finite part for numerator; and as many 9s as there are places in the finite part, followed by as many 0s as there are places in the finite part for denominator.*

Whence the truth for the rule is manifest.

Exercise 72.

1. Reduce $\cdot83\dot{2}\dot{5}$, $\cdot147\dot{6}5\dot{8}$, and $\cdot432\dot{0}07\dot{5}$ to their equivalent vulgar fractions. *Ans.* $\frac{8242}{9900}=\frac{4121}{4950}$, $\frac{147511}{999000}$, and $\frac{4319643}{9999000}=\frac{1439881}{3333000}$.

2. Reduce $875\cdot49\dot{6}\dot{5}$ and $801\cdot82\dot{7}5\dot{6}$ to their equivalent mixed numbers. *Ans.* $875\frac{1229}{2475}$ and $801\frac{1531}{1850}$.

3. Reduce $\cdot08\dot{3}$, $\cdot0\dot{7}1428\dot{5}$, and $\cdot123\dot{4}5\dot{6}$ to their equivalent vulgar fractions. *Ans.* $\frac{1}{12}$, $\frac{1}{14}$, and $\frac{41111}{333000}$.

4. Reduce $\cdot70\dot{8}\dot{4}$, $\cdot964\dot{3}\dot{2}$, $\cdot002\dot{0}\dot{7}$, and $\cdot1432\dot{7}\dot{1}$ to their equivalent vulgar fractions. *Ans.* [illegible], [illegible], $\frac{41}{19800}$, and $\frac{141839}{990000}$.

79. There are several properties belonging to repetends which it is necessary to remember. They are as follows:

1st. Any finite decimal may be regarded as a repetend if we make the 0s recur:

Thus, $\cdot 27 = \cdot 27\dot{0} = \cdot 27\dot{0}\dot{0} = \cdot 27\dot{0}0\dot{0} = \cdot 27\dot{0}000\dot{0}$, &c.

2nd. A repetend having any number of places may be reduced to one having *twice, thrice*, &c., that number of places.

Thus a repetend having 2 places may be reduced to one having 4, 6, 8, 10, 12, &c., places.

For example, $\cdot\dot{3}\dot{7}\dot{2}$...

For example, $\cdot 3\dot{7}\dot{2} = \cdot 3\dot{7}27\dot{2} = \cdot 3\dot{7}2727\dot{2}$, &c.

$\cdot 28\dot{2}13\dot{4} = \cdot 28\dot{2}134213\dot{4} = \cdot 28\dot{2}1342134213\dot{4}$, &c.

3rd. Two or more repetends, having a different number of places in each, may be reduced to others having the same number of places in each, by the following—

RULE.

Take the numbers indicating how many places there are in each repetend, and find their least common multiple. Reduce each repetend to that number of places.

Thus, let it be required to reduce $\cdot 14\dot{7}$, $\cdot 9\dot{3}\dot{2}$, $\cdot 8\dot{4}1\dot{7}$, to repetends having the same number of places.

Here the numbers of places are 1, 2, and 3, and the least common multiple of 1, 2 and 3 is 6, and hence each new repetend must have 6 places.

Therefore $\cdot 14\dot{7} = \cdot 14\dot{7}7777\dot{7}$, $\cdot 9\dot{3}\dot{2} = \cdot 9\dot{3}2323\dot{2}$, and $\cdot 8\dot{4}1\dot{7} = \cdot 8\dot{4}1741\dot{7}$.

4th. Any repetend may be transformed into another having a finite part and an infinite part containing as many places as the original repetend, and hence any two or more repetends may be made similar,

Thus, $\cdot 4\dot{1}2\dot{3} = \cdot 41\dot{2}3\dot{1} = \cdot 412\dot{3}1\dot{2}$, &c.

$7\cdot\dot{6}5432\dot{1} = 7\cdot 6\dot{5}4321\dot{6} = 7\cdot 65\dot{4}3216\dot{5}$, &c.

5th. Having made two or more repetends similar by the last article, they may be made coterminous by the preceding one, and hence two or more repetends may always be made *similar* and *coterminous*.

6th. If several repetends of equal places be added together their sum will be a repetend of the same number of places; since every set of periods will give the same sum.

ADDITION OF CIRCULATING DECIMALS.

80. To add circulating decimals—

RULE.

Make the repetends similar and coterminous and write them under one another, so as to have the units of the same order in the same vertical column.

Add, beginning at the right hand side and carrying what would have been obtained if the decimals had been carried out two or three places further.

Example.—Add together $·78\dot{3}$, $·9\dot{2}\dot{7}$, $·\dot{4}2\dot{1}$ and $9·12\dot{3}45\dot{6}$.

Dissimilar.		Similar.		Similar and Coterminous.
$·78\dot{3}$	=	$·78\dot{3}$	=	$·78\dot{3}33333333333\dot{3}$
$·9\dot{2}\dot{7}$	=	$·92\dot{7}\dot{2}$	=	$·92\dot{7}2727272727\dot{2}$
$·\dot{4}2\dot{1}$	=	$·42\dot{1}4\dot{2}$	=	$·42\dot{1}4214214214\dot{2}$
$9·12\dot{3}45\dot{6}$	=	$9·12\dot{3}45\dot{6}$	=	$9·12\dot{3}4563456345\dot{6}$
				1 carried.
		Sum,	=	$11·25\dot{5}4838276620\dot{4}$

Exercise 73.

1. Add together $·\dot{9}$, $6·3\dot{2}\dot{7}$, $19·43$, $27·02\dot{7}\dot{8}$ and $·0347\dot{1}2\dot{3}$.

Ans. $53·8198\dot{6}3827\dot{4}$.

2. Add together $7·4\dot{2}\dot{7}$, $9·1\dot{2}3\dot{4}$, $17·298\dot{7}64\dot{3}$ and $18·\dot{6}\dot{7}$.

Ans. $52·5202\dot{2}820390147\dot{1}$.

3. Add together $4·\dot{9}\dot{5}$, $7·\dot{1}6\dot{4}$, $4·7\dot{1}2\dot{3}$ and ·9[illegible]. *Ans.* $17·8092\dot{5}0218\dot{8}$.

4. Add together $1·5$, $99·083$, $·\dot{1}6\dot{2}$, $·\dot{8}1\dot{4}$, $2·\dot{9}\dot{3}$, $3·\dot{7}6923\dot{0}$, $97·2\dot{6}$ and $134·\dot{0}\dot{9}$. *Ans.* $339·626\dot{1}7744\dot{3}$.

SUBTRACTION OF CIRCULATING DECIMALS.

81. To subtract one repetend from another—

RULE.

Make the repetends similar and coterminous, and write one beneath the other, so as to have units of the same order in the same vertical column.

Subtract as in whole numbers, taking notice whether one would have been borrowed if the periods had been extended.

EXAMPLE.—From $97{\cdot}034\dot{2}\dot{9}$ take $11{\cdot}03\dot{8}7\dot{6}$.

Dissimilar.	Similar.	Similar and Coterminous.
$97{\cdot}034\dot{2}\dot{9}$	$97{\cdot}034\dot{2}\dot{9}$	$97{\cdot}034\dot{2}9292\dot{9}$
$11{\cdot}03\dot{8}7\dot{6}$	$11{\cdot}038\dot{7}6\dot{8}$	$11{\cdot}038\dot{7}6876\dot{8}$
		True difference, $85{\cdot}995\dot{5}2416\dot{0}$

If the periods had been extended, we would have had to borrow one from the last figure of the minuend period; and bearing this in mind, we say 9 from 9, 0, &c.

EXERCISE 74.

1. From $729{\cdot}\dot{3}42\dot{7}$ take $93{\cdot}126$. *Ans.* $636{\cdot}216\dot{7}4\dot{2}$.
2. From $1{\cdot}4\dot{3}729\dot{1}$ take ${\cdot}007\dot{1}\dot{3}$. *Ans.* $1{\cdot}430\dot{1}60059782\dot{4}$.
3. From $1{\cdot}\dot{2}75\dot{4}$ take ${\cdot}\dot{4}738\dot{4}$. 1.12754 wrong *Ans.* ${\cdot}65\dot{3}7001628090\dot{7}$.
4. From $42{\cdot}1876\dot{8}$ take $17{\cdot}000000\dot{8}43\dot{2}$. *Ans.* $25{\cdot}187682\dot{4}90\dot{0}$.

MULTIPLICATION OF CIRCULATING DECIMALS.

82. To multiply one repetend by another or by a finite decimal—

RULE.

Change the decimals into their equivalent vulgar fractions (Arts. 77 and 78), multiply these together, and reduce the product to its equivalent decimal.

EXAMPLE 1.—Multiply ${\cdot}\dot{3}$ by ${\cdot}\dot{7}\dot{8}$.

$${\cdot}\dot{3}=\tfrac{3}{9}=\tfrac{1}{3} \text{ and } {\cdot}\dot{7}\dot{8}=\tfrac{78}{99}=\tfrac{26}{33}.$$

Therefore, ${\cdot}\dot{3}\times{\cdot}\dot{7}\dot{8}=\frac{1}{3}\times\frac{26}{33}=\frac{26}{99}={\cdot}\dot{2}\dot{6}$ *Ans.*

EXAMPLE 2.—Multiply ${\cdot}3\dot{1}\dot{8}$ by ${\cdot}7\dot{4}3\dot{2}$.

$${\cdot}3\dot{1}\dot{8}=\tfrac{7}{22} \text{ and } {\cdot}7\dot{4}3\dot{2}=\tfrac{55}{74}.$$

Therefore, ${\cdot}3\dot{1}\dot{8}\times{\cdot}7\dot{4}3\dot{2}=\frac{7}{22}\times\frac{55}{74}=\frac{35}{148}={\cdot}23\dot{6}4\dot{8}$.

EXERCISE 75.

1. Multiply $7{\cdot}25$ by $2{\cdot}\dot{9}$. *Ans.* $21{\cdot}75$.
2. Multiply ${\cdot}\dot{2}9\dot{7}$ by $7{\cdot}72$. *Ans.* $2{\cdot}29\dot{5}1\dot{3}$.

3. Multiply $\cdot 8\dot{1}\dot{8}$ by $\cdot 77$. *Ans.* $\cdot 63$.

4. Multiply $1\cdot 7\dot{3}\dot{5}$ by $\cdot 47058$. *Ans.* $\cdot 81654\dot{1}6835\dot{0}$.

5. Multiply $4\cdot 72\dot{2}$ by $\cdot\dot{1}9\dot{8}$. *Ans.* $\cdot\dot{9}3\dot{5}$.

DIVISION OF CIRCULATING DECIMALS.

83. To divide one repetend by another or by a finite decimal—

RULE.

Change the decimals into their equivalent vulgar fractions, divide as in Art. 48, and reduce the result to its corresponding decimal.

EXAMPLE.—Divide $\cdot 4\dot{2}\dot{7}$ by $\cdot 8\dot{1}\dot{8}$.

$$\cdot 4\dot{2}\dot{7}=\tfrac{47}{110} \text{ and } \cdot 8\dot{1}\dot{8}=\tfrac{9}{11}.$$

Therefore, $\cdot 4\dot{2}\dot{7}\div\cdot 8\dot{1}\dot{8}=\frac{47}{110}\div\frac{9}{11}=\frac{47}{110}\times\frac{11}{9}=\frac{47}{90}=0\cdot 5\dot{2}$.

EXERCISE 76.

1. Divide $\cdot\dot{0}8\dot{2}$ by $\cdot\dot{1}2\dot{3}$. *Ans.* $\cdot\dot{6}$.

2. Divide $389\cdot\dot{1}8\dot{5}$ by $15\cdot\dot{7}$. *Ans.* $24\cdot\dot{6}$.

3. Divide $\cdot 81654\dot{1}6835\dot{0}$ by $\cdot 4705\dot{8}$. *Ans.* $1\cdot 7\dot{3}\dot{5}$.

4. Divide $\cdot\dot{4}\dot{5}$ by $\cdot\dot{1}1888\dot{1}$. *Ans.* $3\cdot\dot{8}23529411764705\dot{8}$.

EXERCISE 77.

MISCELLANEOUS EXERCISE ON DECIMALS.

1. Reduce $\frac{1}{2}$ of $\frac{3}{7}$ of $\frac{4}{15}$ of 14 to its equivalent decimal.
2. Multiply $\cdot 6\dot{7}$ by $2\cdot\dot{1}\dot{8}$.
3. Find the value of $\cdot 678125$ of a week.
4. Reduce $\cdot 92\dot{4}3\dot{7}$ to its equivalent fraction.
5. Add together $67\cdot 2\dot{3}\dot{4}$, $96\cdot\dot{7}1\dot{8}$, and $91\cdot 03471\dot{2}3\dot{4}$, and from their sum take $100\cdot 123\dot{4}56789$.
6. Reduce 5 fur. 36 rds. 2 yds. 2 ft. 9 in. to the decimal of a mile.

14 [illegible]

7. Find the difference between $97\cdot 42857\dot{1}$ sq. ft. and $100\cdot\dot{8}$ sq. in.
8. What is the value of $\cdot 917897\dot{7}\dot{2}$ of two acres?

9. Reduce $11{\cdot}2\dot{8}\dot{7}$ and $1{\cdot}0\dot{4}2857\dot{1}$ to vulgar fractions.

10. Divide $47{\cdot}345$ by $1{\cdot}\dot{7}\dot{6}$.

11. From $85{\cdot}\dot{6}\dot{2}$ take $18{\cdot}\dot{7}648\dot{2}$.
12. What is the difference between ·734 of a lb. and ·198 of an oz. avoirdupois?
13. How many yards of carpet 2 ft. $5\frac{1}{3}$ in. wide will be required to cover a floor $27{\cdot}\dot{3}$ ft. long and $20{\cdot}1\dot{6}$ ft. wide?

14. Multiply $3{\cdot}1\dot{4}\dot{5}$ by $4{\cdot}\dot{2}9\dot{7}$.
15. How many finite places are there in the decimals corresponding to $\frac{3}{40}$, $\frac{7}{24}$, $\frac{5}{18}$, $\frac{11}{144}$, $\frac{6}{90}$, and $\frac{112}{3584}$?

16. Add together $81\frac{3}{8}$, $61{\cdot}12\dot{6}$, $328\frac{3}{55}$, and $5{\cdot}\dot{6}2\dot{4}$.

17. Reduce $\left(\dfrac{4{\cdot}\dot{4}-2{\cdot}8\dot{3}}{1{\cdot}\dot{6}+2{\cdot}\dot{6}2\dot{9}} \text{ of } \dfrac{6{\cdot}8 \text{ of } 3}{2{\cdot}25}\right)+\dfrac{2{\cdot}3 \text{ of } 2{\cdot}\dot{2}\dot{7}}{1{\cdot}1\dot{3}\dot{6}}$ to a simple quantity.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*The numbers after the questions refer to the articles of the Section.*

1. What is a decimal fraction? (52)
2. What is the distinction between a decimal and its corresponding decimal fraction? (54 and Art. 47, Sec. I.)
3. How is a decimal reduced to its corresponding decimal fraction? (55)
4. How is a vulgar fraction reduced to a decimal? (56)
5. How would you reduce 4 oz. 17 dwt. 16 grs. to the decimal of a lb.? (57)
6. How would you find the value of ·71845 of a French ell? (58)
7. What is meant by repeating or circulating decimals? (60)
8. What is a repetend, period, or circle? (61)
9. What is a single repetend, and how is it expressed? (62 and 63)
10. What is a circulating decimal or compound repetend, and how is it expressed? (64 and 65)
11. What is a pure repetend? (66)
12. What is a mixed repetend? (67)
13. What are similar repetends? Give an example. (68)
14. What are dissimilar repetends? Give examples. (69)
15. What are coterminous repetends? Give examples. (70)
16. When are repetends said to be both similar and coterminous? Give examples. (71)
17. When can a vulgar fraction be exactly expressed by a decimal? (72)
18. Show that this must necessarily be the case. (72)
19. How can we ascertain the number of places in the finite part of the decimal corresponding to any vulgar fraction? (73)
20. If the decimal corresponding to any vulgar fraction contain a repetend, what is the greatest number of places that repetend can contain? (75)
21. Show that this must necessarily be the case.
22. What are perfect repetends? (76)
23. How is a pure repetend reduced to a vulgar fraction? (77)
24. How is a mixed repetend reduced to a vulgar fraction? (78)
25. Show the truth of this rule. (78)
26. Show that any finite decimal may be made into a repetend. (79)

27. Show that any repetend may be reduced to another having twice, thrice, &c., as many places. (79)
28. Show that any number of repetends may be made to have the same number of places, and give the rule. (79)
29. Show that any pure repetend may be transformed into a mixed repetend? (79)
30. Show that two or more repetends may be made similar and coterminous. (79)
31. How are circulating decimals added? (80)
32. How are circulating decimals subtracted? (81)
33. How do we multiply circulating decimals together? (82)
34. How do we divide one circulating decimal by another? (83)

EXERCISE 78.

MISCELLANEOUS EXERCISE.

(*On preceding Rules.*)

1. Transform 4312131 *quinary*, into the *nonary*, *ternary*, and *octenary* scales, and prove the results by reducing all four numbers to the decimal scale.
2. Write down seven hundred and two trillions seven millions thirty thousand and seventeen, and four millions and seventy-six tenths of quadrillionths.
3. Divide 976·432 by ·00000096.
4. What is the value of $\dfrac{(2\frac{7}{8}+\cdot5625-1\cdot5+\frac{1}{16})\div\frac{11}{8}}{\dfrac{(1\frac{2}{11}\times\frac{4}{9}\times296\times\frac{1}{101}+\frac{11}{8})\div9472947}{\dfrac{19}{6}}}$
5. Divide 97 lb. 3 oz. 4 dr. 1 scr. 17 grs. by 9 lb. 7 oz. 7 dr. 2 scr.
6. A wall is to be built 15 yards long, 7 feet high, and 18 in. thick, with a doorway 6 ft. high and 4 ft. wide; how many bricks will it require, the solid contents of each being 108 cubic inches?
7. Multiply 9 ft. 6′ 4″ 7‴ by 11 ft. 7′ 9″ 11‴.
8. Find the value of $\dfrac{4\frac{3}{7}+\frac{8}{9}-\frac{7}{12}}{\frac{3}{4}\text{ of }\frac{5}{13}+\frac{1}{6}\text{ of }\frac{8}{9}}$.
9. Reduce 782436 pints to bushels, &c.
10. Find the least common multiple of 77, 42, 27, 21, 33, 14, 7, 11, 63, and 30.
11. Divide 36*t*87942 by 28*e*4 in the *duodecimal* scale. Also change 3762814 from the *nonary* to the decimal scale.
12. How many divisors has the number 150528?
13. Find the value of ·1234625 of 2 weeks and 2 days.
14. Multiply 27 lb. 4 oz. 3 dr., avoirdupois, by 728¼.
15. Add together \$98·17, \$42·29, £16 3s. 8¾d., \$97·19, \$127.87½, and from their sum subtract £67 17s. 7¼d.
16. Reduce $\cdot\dot{8}$, $\cdot\dot{7}\dot{6}$, $\cdot9\dot{1}2\dot{8}$, and $\cdot003\dot{8}2\dot{7}$ to their equivalent vulgar fractions.

17. Take the number 704, and by removing the decimal point, (1) Make it 10000 times greater; (2) make it 10000000 times less; (3) make it billions; (4) make it hundredths of billionths; (5) make it tenths of millionths; (6) make it hundredths.

18. Reduce
$$\frac{\dfrac{\left[\left\{\left(2\frac{1}{3}\times\cdot5\text{ of }1\frac{4}{7}\right)+9\frac{1}{2}\right\}+\cdot\dot{0}\dot{9}+\frac{22}{231}\right\}-11\frac{6}{17}\right]\div\left(\frac{1}{5}\text{ of }\cdot1\dot{6}\right)^{*}}{\left[\left(\cdot76\dot{3}276\dot{3}\times11\right)\times\frac{1}{8}\text{ of }\frac{81}{106}\right]\times\left(\frac{1}{2}\text{ of }\cdot2\text{ of }\cdot\dot{3}\text{ of }\cdot25\text{ of }96\right)\div\cdot\dot{2}}}{\frac{1}{4}\text{ of }\cdot67\dot{3}246\dot{7}\div\frac{1}{3}}.$$

19. Divide £550 3s. 1½d. among 4 men, 6 women, and 8 children, giving to each man double of a woman's share; and to each woman triple of a child's.
20. Add together $16\frac{7}{11}$, $19\frac{1}{6}$, $23\frac{7}{8}$, and $129\frac{5}{7}$.
21. Write down all the divisors of 8100.
22. Find the G. C. M. of 2691, 11817 and 9828.
23. Find the exact length of the lunar month which contains 2551443 seconds, and of the solar year, which contains 31556928 seconds.
24. How many times will a carriage wheel turn in going from Toronto to Hamilton, a distance of 38 miles, the circumference of the wheel being 14 feet 11 inches?
25. What is the weight of the water contained in a rectangular cistern 11 feet wide, 13 feet long, and 15 feet deep, and how many gallons of water does it contain?

NOTE.—A cubic foot of water weighs 62·5 lbs. and a gallon weighs 10 lbs.

26. Reduce £73 17s. 11¾d. to dollars and cents.
27. From $93\frac{4}{11}$ take $76\frac{12}{23}$ and divide the result by $\frac{17}{253}$.
28. Find the value of $\dfrac{5\frac{2}{3}\div\frac{2}{3}}{1\frac{1}{3}\text{ of }\frac{2}{3}+10\frac{1}{3}}\times\frac{3}{8}\text{ of }\dfrac{1\frac{1}{2}\text{ of }4\frac{1}{6}}{13\frac{7}{8}\text{ of }5\frac{1}{2}}$.
29. Transform 91342 *undenary* into the *quinary*, *duodenary* and *binary* scales and prove the results by reducing all four numbers to the decimal scale.
30. What are the prime factors of 7680?
31. Reduce 72 miles, 3 fur., 7 per., 2 yds., 1 ft., 7 in. to lines.
32. Find the price of 97 pairs of gloves at 47 cents per pair.
33. What is the worth of a pile of cord wood 73 feet long, 4 feet wide and 11 feet high, at $3·62½ per cord?
34. Divide 93·723 by 29·4173.
35. How many bushels of oats are there in 73429 lbs.?
36. What is the worth of 719630 lbs. of wheat at $1·80 per bushel?
37. Add together $72·14 and $93·76; multiply the sum by 9·47 and divide the product equally among 11 persons.
38. Find the G. C. M. of 21389 and 180781.

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* These questions though apparently difficult are not so in reality—they are designed for exercise in cancelling, and do not require much work.

39. Reduce $\frac{7}{11}$, $\frac{4}{5}$, $\frac{9}{7}$, $\frac{5}{13}$, $\frac{11}{14}$, $\frac{7}{10}$, and $\frac{1}{2}$ to equivalent fractions, having a common denominator.

40. Purchased 17 yards of cotton at 11 cents per yard, 19 yards of ribbon at 87½ cents a yard, 14½ yards of silk at \$2·17 a yard, a parasol \$4·75, a bonnet \$11·50, 67 yards of sheeting at 27 cents a yard, 15 yards of French merino at \$1·37½ a yard, and trimmings \$7·98. Required the amount of my bill.

SECTION V.

RATIO AND PROPORTION.

1. Two numbers having the same unit may be compared with one another in two ways.

1st. By considering *how much greater or less* one is than the other; and

2nd. By considering *how many times* one contains the other.

2. Ratio is the relation which one number bears to another with respect to magnitude, when the numbers are compared by considering, not *how much greater or less* one is than the other, but *how many times* or *parts of a time* one contains the other. Hence:

The ratio of two numbers is the *quotient* arising from the division of one by the other.

Thus the ratio of 18 to 6 is 3, since $18 \div 6 = 3$, the ratio of 7 to 21 is $\frac{1}{3}$, since $7 \div 21 = \frac{7}{21} = \frac{1}{3}$.

3. The ratio of one number to another, when measured with respect to their *difference*, is sometimes called *arithmetical ratio*, to distinguish it from the ratio considered as in (Art. 2), which is called geometrical ratio.

In the following pages, whenever the term ratio is used, geometrical ratio is meant; we shall use the term difference in place of arithmetical ratio.

4. Since ratio simply expresses the quotient arising from the division of one number by another, and since (Art. 66, Sect. II.) we have three ways of indicating division, it follows that we have three ways of expressing the ratio of one number to another.

Thus the ratio of 9 to 4 is expressed either by $9 \div 4$, or by $\frac{9}{4}$, or by $9:4$. The ratio of 7 to 13 is indicated either by $7 \div 13$, or by $\frac{7}{13}$, or by $7:13$.

5. Ratio can exist only between numbers of the same kind.

Thus it is obvious that no comparison with respect to magnitude can be made between 6 *hours* and 11 *pounds*, or between 19 *days* and 16 *miles*, &c., i. e., these numbers are not of the same kind, and therefore no ratio can exist between them.

6. Numbers are of the same kind when they are of the same denomination, or when they have the same unit, or when one can be multiplied so as to exceed the other.

7. The two given numbers which constitute the ratio are called the *terms of the ratio;* when spoken of together they are called a *couplet.*

8. The first term of a couplet is called the *antecedent;* the last term, the *consequent.*

When the ratio is expressed in the form of a fraction, the numerator is the antecedent and the denominator the consequent.

9. Ratio is either *direct* or *inverse, simple* or *compound.*

10. A Direct ratio is that which arises from the division of the antecedent by the consequent.

11. An *Inverse* or *Inverted Ratio* is that which arises from the division of the consequent by the antecedent.

Thus the inverse ratio of 15 to 8 is 8 : 15 or $\frac{8}{15}$, or 8÷15, or $\frac{8}{15}$.

12. An Inverse Ratio is sometimes called a reciprocal ratio.

Thus the reciprocal ratio of 15 to 8 is 8 : 15 or $\frac{8}{15}=\frac{1}{\frac{15}{8}}$=inverse ratio of 15 to 8.

13. The reciprocal of a quantity is unity divided by that quantity.

Thus the reciprocal of 8 is $\frac{1}{8}$; of 11, $\frac{1}{11}$; of $\frac{4}{5}$, $\frac{5}{4}$; $\frac{8}{15}$, $\frac{15}{8}$; of $\frac{1}{9}$, 9; of $\frac{4}{13}$, $\frac{13}{4}$, &c.

14. *When the direct ratio of two numbers is expressed by points, the inverse or reciprocal ratio is expressed by inverting the order of the terms; when by a fraction, by inverting the fraction.*

15. A Simple Ratio is one that has but *one* antecedent and *one* consequent.

Thus 9 : 3, 7 : 11, 18 : 2, &c., are simple ratios.

16. A Compound Ratio is a ratio produced by *compounding* or multiplying together the corresponding terms of two or more simple ratios.

Thus, the simple ratio of.................. 9 : 3 is 3.
the simple ratio of.................. 24 : 2 is 12.
The ratio compounded of these is 216 : 6=36.

17. It must be distinctly remembered that a compound ratio is of the same nature as any other ratio, and, like a simple ratio, consists of one antecedent and one consequent. The term compound ratio is used merely to indicate the *origin* of the ratio in particular cases.

18. *Ratios are compounded by multiplying together all the antecedents for a new antecedent, and all the consequents for a new consequent.*

Thus, the ratios compounded of 2 : 7, 2 : 3, 5 : 11, and 4 : 3 is 2×2×5×4 : 7×3×11×3 or 80 : 693.

EXERCISE 79.

1. What is the ratio of 27 to 3? *Ans.* 9.
2. What is the ratio of 7 to 11? *Ans.* $\frac{7}{11}$.
3. What is the ratio of 9 to 27? *Ans.* $\frac{1}{3}$.
4. What is the ratio of 42 to 5? *Ans.* 8$\frac{2}{5}$.
5. What is the ratio of 72 to 6? *Ans.* 12.

Required the ratio of the following numbers:—

6. 5 to 25. *Ans.* $\frac{1}{5}$.
7. 49 to 7. *Ans.* 7.
8. 83 to 7. *Ans.* 11$\frac{6}{7}$.
9. 187 to 11. *Ans.* 17.
10. 19 to 152.
11. 23 to 299.
12. 147 to 21.
13. $17 to $8·50. *Ans.* 2.
14. $93 to $31. *Ans.* 3.
15. 14 bus. to 2 pks. *Ans.* 28.
16. 40 m. to 12 fur. *Ans.* 26$\frac{2}{3}$.
17. 24 lb. to 12 oz.
18. 17 shillings to £51.
19. 16 acres to 30 sq. per.

Required the inverse ratio of the following numbers:

20. 7 to 21. *Ans.* 3.
21. 12 to 2. *Ans.* $\frac{1}{6}$.
22. 27 to 6. *Ans.* $\frac{2}{9}$.
23. 9 to 36. *Ans.* 4.
24. 19 to 57.
25. 81 to 9.
26. 187 to 17.
27. 6 days to 4 weeks. *Ans.* 4$\frac{2}{3}$.
28. 11 min. to 30 sec. *Ans.* $\frac{1}{22}$.
29. 4 lbs. to 12 oz. Av. *Ans.* $\frac{3}{16}$.
30. 3 qts. to 43 gals. *Ans.* 57$\frac{1}{3}$.
31. 70 per. to 2 miles.
32. 7 Flem. ells to 9 Eng. ells.
33. 11 oz. to 68 scruples.

Required the reciprocal ratio of the following numbers:—

34. 7 to 42. *Ans.* $\frac{1}{7} : \frac{1}{42} = 6$.
35. $\frac{1}{8}$ to $\frac{1}{2}$. *Ans.* 8 : 2 = 4.
36. 42 to 28. *Ans.* $\frac{1}{42} : \frac{1}{28} = \frac{2}{3}$.
37. 17 to 68.
38. 19 to 17.
39. $\frac{1}{24}$ to $\frac{1}{36}$. *Ans.* $\frac{3}{2}$.
40. 72 to 18. *Ans.* $\frac{1}{4}$.
41. 512 to 32. *Ans.* $\frac{1}{16}$.
42. $\frac{1}{4}$ to $\frac{7}{8}$.
43. $\frac{2}{3}$ to $\frac{4}{5}$.

Required the ratios compounded of the following ratios:—

44. 2 to 3, 5 to 7 and 1 to 7. *Ans.* 10 to 147.
45. 8 to 6 and 17 to 3. *Ans.* 136 to 18.
46. 9 to 8, 7 to 6, 5 to 6, 4 to 3 and 2 to 1. *Ans.* 2520 : 864.
47. 1 to 7, 1 to 3, 3 to 1 and 5 to 1. *Ans.* 15 : 21.
48. 2 to 5, 3 to 7, 4 to 5, 21 to 2 and 1 to 9. *Ans.* 504 : 3150.

19. Since the antecedent of a couplet is a dividend, the consequent a divisor, and the ratio the quotient, it follows from the principles established in Arts. 79–84, Sect. II., that;—

1st. Multiplying the antecedent of a couplet or dividing the consequent by any number multiplies the ratio by that number.

Thus the ratio of 28 to 112 = $\frac{1}{4}$.
The ratio of 28 × 3 to 112 = $\frac{3}{4}$ = $\frac{1}{4}$ × 3 = three times the ratio of 28 to 112.

2nd. Dividing the antecedent of a couplet or multiplying the consequent by any number divides the ratio by that number.

Thus the ratio of 64 to 16 = 4.
The ratio of 64 ÷ 2 to 16 = 32 : 16 = 2 = 4 ÷ 2 = half the ratio of 64 to 16.

3rd. Multiplying or dividing both antecedent and consequent of a couplet by the same number does not alter the value of the ratio.

Thus the ratio of 18 to 6 is 3.
The ratio of 18 × 7 : 6 × 7 = 126 : 42 = 3 = ratio of 18 ÷ 2 : 6 ÷ 2 = 9 : 3.

20. Since any number of ratios to be compounded together may be expressed as fractions and then compounded by the rule for multiplication of fractions (Art. 45, Sect. IV.) it follows that :—

When several ratios are to be compounded together, we may, before multiplying the corresponding terms together, cancel any factor that is common to an antecedent and a consequent.

Example 1.—Compound together 4 : 17, 34 : 55, 11 : 2, 13 : 7, and 21 : 65.

OPERATION.

$$\left.\begin{array}{rcl} 4 & : & \cancel{17} \\ \overset{\cancel{2}}{\cancel{34}} & : & \overset{5}{\cancel{55}} \\ \cancel{11} & : & \cancel{2} \\ \cancel{13} & : & \cancel{7} \\ \overset{3}{\cancel{21}} & : & \overset{5}{\cancel{65}} \end{array}\right\} = 4 \times 3 : 5 \times 5 \quad \text{or} \quad 12 : 25 \textit{ Ans.}$$

Explanation.—17 cancels 17 and reduces 34 to 2 and this 2 cancels 2, the third consequent; 11 reduces 55 to 5; 13 reduces 65 to 5 and 7 reduces 21 to 3. The only antecedents now left are 4 and 3 which multiplied together make 12, and the only remaining consequents are 5 and 5 which multiplied together make 25. The ratio 12 to 25 is therefore the ratio compounded of all the given ratios.

Example 2.—Compound the following ratios :—

OPERATION.

$$\left.\begin{array}{rcl} \cancel{7} & : & \overset{\cancel{2}}{\cancel{16}} \\ \overset{\cancel{3}}{\cancel{24}} & : & \cancel{3} \\ 9 & : & \overset{\cancel{7}}{\cancel{49}} \\ \overset{\cancel{2}}{\cancel{14}} & : & \cancel{11} \\ \overset{2}{\cancel{22}} & : & 13 \end{array}\right\} = 9 \times 2 : 13 \quad \text{or} \quad 18 : 13 \textit{ Ans.}$$

Example 3.—Find the ratio compounded of the following ratios :—

OPERATION.

$$\left.\begin{array}{rcl} 1 & : & \cancel{7} \\ \cancel{16} & : & \cancel{23} \\ \cancel{29} & : & \cancel{14} \\ \overset{\cancel{7}}{\cancel{98}} & : & \overset{\cancel{29}}{\cancel{319}} \\ \overset{\cancel{11}}{\cancel{253}} & : & \overset{4}{\cancel{64}} \end{array}\right\} = 1 : 4 \textit{ Ans.}$$

EXERCISE 80.

1. Find the ratio compounded of 9 : 16, 25 : 31, 341 : 18 and 48 : 100. *Ans.* 33 : 8.
2. Find the ratio compounded of 18 : 25, 7 : 9, 11 : 12, and 91 : 49. *Ans.* 143 : 150.
3. Find the ratio compounded of 1 : 2, 2 : 3, 3 : 4, 4 : 5, 5 : 6 and 7 : 11. *Ans.* 7 : 66.
4. Find the ratio compounded of 2 : 5, 8 : 11, 14 : 17 and 187 : 112. *Ans.* 2 : 5.
5. Find the ratio compounded of 3 : 5, 7 : 9, 11 : 13, 15 : 17 and 19 : 21. *Ans.* 209 : 663.

21. If the antecedent of a couplet be *equal* to the consequent, the ratio is equal to 1 and is called a *ratio of equality.*

If the antecedent be greater than the consequent the ratio is greater than 1 and is called a *ratio of greater inequality.*

If the antecedent be less than the consequent the ratio is less than 1, and is called a *ratio of less inequality.*

Thus the ratio of 7 : 7 = 1 is a ratio of equality.
The ratio of 7 : 2 = $3\frac{1}{2}$ is a ratio of greater inequality.
The ratio of 7 14 = $\frac{1}{2}$ is a ratio of less inequality.

EXERCISE 81.

In examples 1–43 of Exercise 79 point out which are ratios of *greater* and which ratios of *less inequality.*

22. *Ratios are compared with one another by expressing them in the form of fractions—reducing these to their equivalent fractions having a common denominator and comparing the numerators.*

Ratios may also be compared by actually dividing the antecedent by the consequent and thus ascertaining which gives the greatest quotient.

NOTE.—The latter method is usually the more convenient.

EXAMPLE 1.—Which is the greatest and which the least of the following ratios, viz: 3 : 4, 7 : 8, and 9 : 10?

$$\text{By 1st Rule}\quad \left.\begin{array}{l} 3 : 4 = \frac{3}{4} = \frac{30}{40} \\ 7 : 8 = \frac{7}{8} = \frac{35}{40} \\ 9 : 10 = \frac{9}{10} = \frac{36}{40} \end{array}\right\} \text{Hence } 9 : 10 \text{ is greatest and } 3 : 4 \text{ least.}$$

$$\text{By 2nd Rule}\quad \left.\begin{array}{l} 3 : 4 = 3 \div 4 = \cdot 75 \\ 7 : 8 = 7 \div 8 = \cdot 875 \\ 9 \cdot 10 = 9 \div 10 = \cdot 9 \end{array}\right\} \text{Hence } 9 : 10 \text{ is greatest and } 3 : 4 \text{ least.}$$

EXAMPLE 2.—Compare together the following ratios, 7 : 8, 2 : 3 and 11 : 13 and 5 : 6.

By 1st Rule
$7 : 8 = \frac{7}{8} = \frac{819}{936}$
$2 : 3 = \frac{2}{3} = \frac{624}{936}$
$11 : 13 = \frac{11}{13} = \frac{792}{936}$
$5 : 6 = \frac{5}{6} = \frac{780}{936}$
Hence 7 : 8 is the greatest and 2 : 3 is the least.

By 2nd Method
$7 : 8 = 7 \div 8 = \cdot 875$
$2 : 3 = 2 \div 3 = \cdot\dot{6}$
$11 : 13 = 11 \div 13 = \cdot\dot{8}4615\dot{3}$
$5 : 6 = 5 \div 6 = \cdot 8\dot{3}$
Hence 7 : 8 is the greatest and 2 : 3 the least.

EXERCISE 82.

1. Point out which is greatest and which least of the ratios 7 : 4, 6 : 3, 17 : 8, and 11 : 5.
Ans. 11 : 5 is greatest and 7 : 4 least.
2. Point out which is greatest and which least of the ratios 16 : 9, 10 : 3, 7 : 2, and 8 : 3. *Ans.* 7 : 2 is greatest and 16 : 9 least.
3. Point out which is greatest and which least of the ratios 7 : 33, 11 : 49, 16 : 71, and 21 : 106.
Ans. 16 : 71 is the greatest and 21 : 106 least.

23. If the terms of two or more couplets, having the same ratio, be added together, the resulting couplet will have the same ratio.

Thus, the ratio of 6 : 2=3, the ratio of 21 : 7=3, and the ratio of 33 : 11=3, and the ratio 6+21+33 to 2+7+11, that is, of 60 to 20 is also 3.
That is, if 6 : 2=21 : 7=33 : 11, then 6+21+33 : 2+7+11=6 : 2.

24. If from the terms of any couplet the terms of another couplet having the *same ratio* be subtracted, then the resulting couplet will have the rame ratio.

Thus, the ratio of 35 to 5 is 7, and the ratio of 14 to 2 is 7. So also the ratio of 35−14 : 5−2, that is, of 21 : 3 is 7, or, if 35 : 5=14 : 2, then 35−14 : 5−2= 35 : 5.

25. A ratio of *greater inequality* is *diminished* by adding the same number to both terms.

Thus, the ratio of 48 : 8=6.
The ratio of 48+12 : 8+12 or 60 : 20=3 which is less than ratio 48 : 8.

26. A ratio of *less inequality* is *increased* by adding the same number to both terms.

Thus, the ratio of 8 : 48=$\frac{1}{6}$.
The ratio of 8+12 : 48+12 or 20 : 60=$\frac{1}{3}$ which is greater than ratio of 8 : 48.

PROPORTION.

27. Proportion is an equality of ratios.

Thus, the ratios 15 : 3 and 25 : 5 constitute a proportion, since 15 : 3 = 5 = 25 : 5.

28. The terms of the two couplets are called proportionals.

29. Proportion may be expressed in two ways,

1st. By placing =, the sign of equality, between the ratios.

2nd. By placing four points, thus : : , between the two ratios.

Thus, we may express the proportion existing between 15, 3, 25, and 5 by 15 : 3 = 25 5, or by 15 : 3 : : 25 : 5.

We read either of them by saying the ratio of 15 to 3 equals the ratio of 25 to 5; or simply 15 is to 3 as 25 is to 5.

NOTE.—The sign : : is supposed to be derived from =, the sign of equality, the four *points* being merely the *extremities* of the lines.

30. In every proportion there must be *four terms*, since there must be two couplets, and each couplet consists of two terms.

31. When three numbers constitute a proportion, one of them is repeated so as to form two terms.

Thus, if 18, 6, and 2 are proportionals.
18 : 6 : : 6 : 2.

In this case the 6, i. e., the term repeated, is called the *middle* term or a mean proportional between the other two numbers.

The 2 is called the *third* term or a *third proportional* to the other two numbers.

32. It is important to remember the distinction between *ratio* and *proportion*.

A ratio consists of two *terms*, an antecedent and a consequent.

A proportion consists of two couplets or four terms.

One ratio may be greater or less than another

One proportion cannot be greater or less than another, since equality does not admit of degrees.

33. The outer terms of a proportion are called the *extremes*, and the two intermediate ones, the *means*.

Thus in the proportion 3 : 17 : : 21 : 119.
3 and 119 are the extremes.
17 and 21 are the means.

34. If four quantities be proportionals, the product of the extremes is equal to the product of the means.

6 : 11 : : 18 : 33. Then 6 × 33 = 11 × 18

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This may be established in the following manner :—6 : 11 = $\frac{6}{11}$ and 18 : 33 = $\frac{18}{33}$, and since 6 : 11 : : 18 : 33, $\frac{6}{11} = \frac{18}{33}$ (Art. 27). Now, since multiplying equals by the same number does not destroy their equality, if we multiply these fractions by 11 we get $6 = \frac{18 \times 11}{33}$; and multiplying each of these by 33, we have 6 × 33 = 18 × 11; but 6 and 33 are the extremes, and 18 and 11 are the means; therefore in any geometrical proportion the product of the extremes equals the product of the means.

The same fact may be established more generally as follows :—

Let a, b, c and d be any four proportionals whatever.
Then $a : b : : c : d$

But $a : b = \frac{a}{b}$ and $c : d = \frac{c}{d}$

Therefore $\frac{a}{b} = \frac{c}{d}$ — Multiplying each of these equals by $b \times d$, we have $a \times d = b \times c$. But a and d are the extremes and b and c are the means, Therefore, &c.

35. This principle then may be considered the *test* of a geometrical proportion. If the product of the extremes equals the product of the means, the four quantities are proportional; if the products are not equal, the numbers are not proportional.

36. It follows from Art. 34 that :—

1st. *If the product of the means be divided by one extreme, the quotient will be the other extreme.*

2nd. *If the product of the extremes be divided by one mean, the quotient will be the other mean.*

and hence,

3rd. *If any three terms of a proportion be given, the fourth may be found thus :*

$$\textit{1st term} = \frac{\textit{2nd term} \times \textit{3rd term}}{\textit{4th term.}}$$

$$\textit{2nd term} = \frac{\textit{1st term} \times \textit{4th term}}{\textit{3rd term.}}$$

$$\textit{3rd term} = \frac{\textit{1st term} \times \textit{4th term}}{\textit{2nd term.}}$$

$$\textit{4th term} = \frac{\textit{2nd term} \times \textit{3rd term}}{\textit{1st term.}}$$

Example 1.—What is the fourth proportional to 7, 11 and 35?

$$\text{4th term} = \frac{\text{2nd term} \times \text{3rd term}}{\text{1st term.}} = \frac{11 \times 35}{7} = 55 \textit{ Ans.}$$

Example 2.—The first, second and fourth terms of a proportion are 9, 16 and 128. Required the third term.

$$\text{3rd term} = \frac{\text{1st} \times \text{4th}}{\text{2nd}} = \frac{9 \times 128}{16} = 72 \textit{ Ans.}$$

Exercise 83.

1. The second, third and fourth terms of a proportion are 17, 11 and 93½. What is the first term? *Ans.* 2.
2. The first, third and fourth terms of a proportion are 21, 63 and 39. Required the second term. *Ans.* 13.
3. The first three terms of a proportion are 2, 3 and 7. What is the fourth term? *Ans.* 10½.
4. The last three terms of a proportion are 91, 88 and 104. Required the first term. *Ans.* 77.

Find the fourth proportional to

5. 4 yds. 18 yds. and $96. *Ans.* $432.
6. 5 lb. 2 lb. and $3·75. *Ans.* $1·50.
7. 1 cwt. 215 cwt. and $7·50. *Ans.* $1612·50.
8. 6 miles, 1 mile and 27 shillings. *Ans.* 4s. 6d.
9. 10 lb. 150 lb. and £6 3s. 9d. *Ans.* £92 16s. 3d.
10. 4 days, 27 days and $100. *Ans.* $675.

37. It will be useful to remember the following properties of a Geometrical proportion. As the proofs are given in every common work on Algebra, it has not been thought advisable to insert them here; *a*, *b*, *c* and *d* stand for any four proportionals whatever.

If $a : b :: c : d$ — Or if $15 : 6 :: 10 : 4$
Alternately $a : c :: b : d$ — $15 : 10 :: 6 : 4$
Inversely $b : a :: d : c$ — $6 : 15 :: 4 : 10$
By Composition $a + b : b :: c + d : d$ — $15 + 6 : 6 :: 10 + 4 : 4$, or $21 : 6 :: 14 : 4$
By Division $a - b : b :: c - d : d$ — $15 - 6 : 6 :: 10 - 4 : 4$, or $9 : 6 :: 6 : 4$
By Conversion $a : a + b :: c : c + d$ — $15 : 15 + 6 :: 10 : 10 + 4$, or $15 : 21 :: 10 : 14$
Or $a : a - b :: c : c - d$ — $15 : 15 - 6 :: 10 : 15 - 4$, or $15 : 9 :: 10 : 6$

38. Proportion in Arithmetic is usually divided into simple, compound and conjoined.

SIMPLE PROPORTION.

39. Simple proportion is frequently called the Rule of Three, because when *three* terms are given, by means of them a fourth may be found. It is also sometimes called the Golden Rule from its extensive utility.

40. Example.—If 16 barrels of flour cost $112, what will 129 barrels cost?

In this and every other question in Simple Proportion there are two ratios, one of which is perfect (*i. e.* has both terms given) and the other imperfect, and from the nature of proportion we know that these two ratios must be both of the same kind, that is, they must be both ratios of *greater inequality* or both ratios of *less inequality*.

Now in the above example, the ratio of $112 to the answer is a ratio of *less inequality* since it is evident that, if 16 barrels cost $112, 129 barrels will cost more. Therefore the other ratio is also a ratio of *less inequality* and must be written 16 : 129.

And since the ratios are equal

$$\begin{array}{c}\text{barrels.} \quad \text{dollars.}\\ 16 : 129 : : 112 : \textit{Ans.}\end{array}$$

Also (Art. 36) $Ans.=\frac{112\times129}{16}=\$903.$

Proof.—Set 903 in the fourth place, thus:

$$16 : 129 : : 112 : 903$$

and see if the product of extremes=product of means (Art. 35.)

$$16\times903=14448=129\times112.$$

From the preceding illustrations and principles we deduce for Simple Proportion the following general

RULE.

Set the given term of the imperfect ratio in the third place, and the letter x, to represent the answer, in the fourth.

Then, if, by the nature of the question, the ratio of the third term to the answer is a ratio of greater inequality, make the remaining ratio a ratio of greater inequality also; but if the ratio of the third term to the answer be a ratio of less inequality, make the other ratio a ratio of less inequality also.

Lastly, (*Art.* 36,) *multiply the second and third terms together, divide the product by the first term, and the quotient will be the answer in the same denomination as the third term.*

Proof.—*Multiply the first term and the answer together, and, if the product is equal to the product of the second and third terms, the work is correct.* (*Art.* 35.)

Example 1.—If a man can walk 155 miles in 12 days, how many miles can he walk in 60 days?

Here the imperfect ratio is 155 miles to x, and, in order to ascertain whether it is a ratio of greater or less inequality, we have merely to ask the following simple question: If a man can walk 155 miles in 12 days, can he walk more or less in 60 days? Evidently more. Therefore the ratio of 155 : x is a ratio of less inequality, or, in other words, the antecedent must be the least of the two numbers, and the statement is

$$\begin{array}{c}\text{days.} \quad \text{miles.}\\ 12 : 60 : : 155 : x.\end{array}$$

Whence the answer $=\frac{60\times155}{12}=775$ miles.

41. Since the second and third terms multiplied together, constitute a dividend, and the first term is a divisor, it is manifest, from the principles of division (Arts. 79–84, Sect. II.), that we may cancel any factor that is common to the first term and either of the other terms.

Thus in the last example we have $12 : 60 : : 155 : x$ and, dividing the first and second by 12, we get $1 : 5 : : 155 : x$ and $155\times5=775$ *Ans.*

Example 2.—If 96 bushels of wheat cost $128, what will 15 bushels cost?

As the answer to the question must be in dollars, the imperfect ratio is $128 : x, and from the nature of the question, we know that 15 bushels will cost less than 96 bushels; we therefore place 15, the smaller of the remaining terms, in the *second place*, and the other term, 96, in the *first place*. Hence the statement is 96 : 15 bushels : : $128 : x.

OPERATION.

$$\begin{array}{llll} \text{bush.} & & \$ & \\ \cancel{96} : & \cancel{15} : : & \cancel{128} : & x \\ \cancel{3} & 5 \times & 4 = & \$20 \textit{ Ans.} \end{array}$$

Here 32 reduces 96 to 3 and 128 to 4, and 3 cancels 3 and reduces 15 to 5.

The teacher would do well to insist upon his pupils performing all questions in Proportion by analysis.

Thus, to solve the last question, we begin as follows: If 96 bushels cost $128, 1 bushel will cost $\frac{1}{96}$ of $128, or $1·33⅓. Then if 1 bushel cost $1·33⅓, 15 bushels will cost 15 times as much, which is $20.

EXAMPLE 3.—If 27 men can mow 60 acres of grass in a day, how many acres can 93 men mow?

OPERATION.

$$\begin{array}{llll} \text{men.} & & \text{acres.} & \\ \cancel{27} : & \cancel{93} : : & \cancel{60} : & x \\ \cancel{9} & 31 & 20 & \\ 3 & & & \end{array}$$

$$\frac{31 \times 20}{3} = 206\frac{2}{3} \text{ acres } \textit{Ans.}$$

Here the imperfect ratio is 60 : x acres, and since 93 men will evidently mow more than 27 men, we make 93 the *second term* and 27 the *first*. Hence the statement is 27 : 93 : : 60 : x. Then 3 reduces 27 to 9 and 93 to 31, and 3 again reduces 9 to 3 and 60 to 20, and the answer is equal to 31 multiplied by 20, and divided by 3.

This question may be performed thus by analysis:

If 27 men mow 60 acres a day, 1 man will mow $\frac{1}{27}$ of 60 acres, or $2\frac{2}{9}$ acres; 93 men will therefore mow 93 times $2\frac{2}{9}$ acres$=206\frac{2}{3}$ *Ans.*

EXERCISE 84.

1. If 11 baskets of peaches cost $13·42, what will 87 baskets cost? *Ans.* $106·14.
2. If 28 cords of wood cost $266, what will 25 cords cost? *Ans.* $237·50.
3. If a man receives $29·20 for 16 days' work, for how many days should he work for $83·60? *Ans.* $45\frac{59}{73}$ days.
4. If 16 bags of potatoes are sold for $12·80, what will 156 bags bring? *Ans.* $124·80.
5. If a stick 7 feet long cast a shadow of 5 feet, what will be the height of a tree which casts a shadow of 112 feet long? *Ans.* $156\frac{4}{5}$ feet.
6. If a stack of hay will feed 27 cows for 99 days, how long will it feed 55 cows? *Ans.* $48\frac{3}{5}$ days.
7. If 9 bushels of peas sow 5 acres, how many bushels will be required to sow 48 acres? *Ans.* $86\frac{2}{5}$ bushels.
8. If 3 men put up 73 perches of fencing in 2 days, how long will they take to put up 803 perches? *Ans.* 22 days.
9. If 176 pails of maple sap make 100 lbs. of sugar, how much sugar will 1128 pails make? *Ans.* $640\frac{10}{11}$ lbs.
10. If it cost $20·88 to weave 108 yards of cloth, what will it cost to weave 465 yards? *Ans.* $89·90.

11. If \$16 pay for the carriage of 72 barrels of flour, for the carriage of how many barrels will \$1278 pay? *Ans.* 5751 barrels.
12. If 11 men plough 165 acres in a week, how many acres would 3 men plough in the same time? *Ans.* 45 acres.
13. If 4 barrels of flour make 250 four-pound loaves of bread, how many such loaves will 67 barrels make? *Ans.* 4187½ loaves.
14. If 190 bushels of apples make 16 barrels of cider, how many barrels of cider will 38 bushels of apples make? *Ans.* 3⅕ barrels.
15. If 90 men can build a wall in 12 days, how many men could build it in 15 days? *Ans.* 72 men.
16. If 17 days' work pay for two barrels of flour, for how many barrels will 279 days' work pay? *Ans.* 32$\frac{14}{17}$ barrels.
17. If a train travel 27 miles per hour, how far will it travel in 24 hours? *Ans.* 648 miles.
18. If 7 cows make 30 lbs. of butter a week, how much may be expected from [illegible] cows? *Ans.* 98$\frac{4}{7}$ lbs.

42. *If any of the terms contain fractions or mixed numbers, apply the rules in Section IV.*

Example 1.—If $\frac{2}{5}$ of a basket of peaches cost $\frac{2}{7}$ of a dollar, how much will $\frac{3}{11}$ of a basket of peaches cost?

OPERATION.

$\frac{2}{5} : \frac{3}{11} :: \frac{2}{7} : x$. Therefore answer $= \frac{2}{7} \times \frac{3}{11} \div \frac{2}{5} = \$\frac{2}{7} \times \frac{3}{11} \times \frac{5}{2} = 19\frac{37}{77}$ cents.

Example 2.—If $\frac{9}{16}$ of a bushel cost $\frac{4}{11}$ of a pound, what will $\frac{11}{12}$ of a bushel cost?

OPERTAION.

$\frac{9}{16} : \frac{11}{12} :: £\frac{4}{11} : x$. Therefore answer $= \frac{4}{11} \times \frac{11}{12} \div \frac{9}{16} = \frac{4}{11} \times \frac{11}{12} \times \frac{16}{9}$ $= £\frac{16}{27} =$ 11s. 10$\frac{2}{9}$d.

Note.—If the first term be a fraction, invert it and connect it to the others by the sign of multiplication.

Exercise 85.

1. If $\frac{3}{16}$ of a ship cost \$9750, what will $\frac{21}{26}$ cost? *Ans.* \$42000.
2. How much will ¼ of a yard come to if ⅞ of a yard cost ⅚ of a shilling? *Ans.* 2$\frac{6}{7}$d.
3. If \$7·49 pay for $\frac{7}{9}$ of a ton of coals, what will 8⅓ tons cost? *Ans.* \$80·25.
4. If 5⅘ yards of broadcloth cost \$28·42, what will $\frac{4}{7}$ of a yard come to? *Ans.* \$2·80.
5. If $\frac{12}{25}$ of a dollar pay for ⅘ of a bag of apples, for what part of a bag will $\frac{7}{20}$ of a dollar pay? *Ans.* $\frac{7}{12}$ of a bag.
6. If \$100 stock is worth \$98⅞, what will \$472$\frac{11}{25}$ stock be worth? *Ans.* \$467·12½.

7. If $17\frac{3}{5}$ tons of hay last a certain number of horses $107\frac{3}{11}$ days, how many days will $11\frac{11}{17}$ tons last the same number of horses? *Ans.* $70\frac{185}{187}$ days.

8. If $22\frac{4}{9}$ cords of wood last as long as $15\frac{7}{13}$ tons of coal, how many cords of wood will last as long as $11\frac{9}{26}$ tons of coal? *Ans.* $16\frac{7}{18}$ cords of wood.

9. If $\frac{1}{2}$ of $\frac{3}{5}$ of $3\frac{1}{3}$ yards of broadcloth cost $\frac{2}{7}$ of $\frac{3}{11}$ of \$$4\frac{2}{3}$, what will $\frac{3}{8}$ of $\frac{1}{2}$ of $\frac{55}{56}$ of a yard cost? *Ans.* $\frac{15}{224}$ or \$0·0669.

43. When the first and second terms are not of the same denomination or contain different denominations—

RULE.

Reduce both to the lowest denomination contained in either, and then apply the rule in Art. 40.

EXAMPLE.—If 11 bushels 2 pks. 1 gal. cost \$74, what will 76 bushels 1 pk. 1 gal. 1 qt. 1 pt. cost?

OPERATION.

The lowest denomination contained in either is pints.

11 bush. 2 pks. 1 gal. : 76 bush. 1 pk. 1 gal. 1 qt. 1 pt. : : \$74 : x; this reduced becomes 744 : 4891 : : \$74 : x.

$$\textit{Ans.}\ \frac{\$74 \times 4891}{744} = \$486{\cdot}47 +$$

In this example 11 bush. 2 pks. 1 gal. = 744 pints and 76 bush. 1 pk. 1 gal. 1 qt. 1 pt. = 4891 pints.

EXERCISE 86.

1. What will 37 sq. yds. 4 ft. 120 in. of painting cost, if 9 sq. yds. 2 ft. cost \$3·50? *Ans.* \$14·245.
2. How much will 12 lb. 10 oz. of silver come to at \$1·25 per oz.? *Ans.* \$192·50.
3. If 10 yards of ribbon cost \$3·40, what will 3 yds. 2 qrs. cost? *Ans.* \$1·19.
4. If 15 oz. 12 dwt. 16 grs. cost \$3·80, what will 13 oz. 14 grs. cost.? *Ans.* \$3·167.
5. What will 3 lb. 1 oz. 11 dwt. cost, if 12 lb. 6 oz. 4 dwt. cost \$600? *Ans.* \$150.
6. If a man can pump 54 barrels of water in 2 hrs. 46 min. 30 sec., in what time will he pump 24 barrels? *Ans.* 1 h. 14 min.
7. What will 73 yds. 3 qrs. 2 na. 1 in. of velvet cost, if 3 Flem. ells 2 qrs. 1 na. cost £4 17s. $8\frac{1}{4}$d.? *Ans.* £128 6s. $10\frac{61}{90}$d.
8. If $4\frac{5}{9}$ oz. avoirdupois cost $8\frac{31}{32}$ shillings, what will $8\frac{13}{24}$ lbs. cost? *Ans.* £13 9s. $0\frac{3}{4}$d.
9. In the copy of a work containing 327 pages, a remarkable passage commences at the end of the 156th page. On what page might it be expected to begin in a copy containing 400 pages? *Ans.* On the 191st page.

10. If the rent of 46 acres, 3 roods, and 14 perches be £100, what will be the rent of 35 acres, 2 roods, and 10 perches?
Ans. £75 18s. $6\frac{2166}{3747}$d.

11. When A had travelled 68 days at the rate of 12 miles a day, B, who had travelled 48 days, overtook him. How many miles a day did B travel, allowing both to have started from the same place?
Ans. 17.

12. If $21\frac{1}{3}$ shillings pay for $16\frac{1}{7}$ lbs. of prunes, how many pounds can be bought for $32\frac{2}{7}$ shillings?
Ans. $24\frac{675}{1568}$ lbs.

13. A ton of coal yields about 9000 cubic feet of gas; a street lamp consumes about 5, and an argand burner (one in which the air passes through the centre of the flame) 4 cubic feet in an hour. How many tons of coal would be required to keep 17493 street lamps and 192724 argand burners in shops, &c., lighted for 1000 hours?
Ans. $95373\frac{4}{9}$.

14. The gas consumed in London requires about 50000 tons of coal per annum. For how long a time would the gas this quantity may be supposed to produce (at the rate of 9000 cubic feet per ton), keep one argand light, (consuming 4 cubic feet per hour) constantly burning?
Ans. 12842 years and 170 days.

15. Suppose 11270 lbs. of beef for a ship's use were to be cut up in pieces of 4 lb., 3 lb., 2 lb., 1 lb., and ½ lb.—there being an equal number of each. How many pieces would there be of each?
Ans. 1073; and 3½ lb. left.

16. The sloth does not advance more than 100 yards in a day. How long would it take to crawl from Toronto to Kingston, allowing the distance to be 180 miles?
Ans. 3168 days, or about 8⅔ years.

17. Suppose that a greyhound makes 27 springs while a hare makes 25, and that their springs are of equal length. How many springs must the hound make to overtake the hare, if the latter has a start of 50 springs?
Ans. 675.

COMPOUND PROPORTION.

44. Compound Proportion is an equality between a *compound* ratio and a *simple* ratio.

Thus 7 : 11 compounded with 22 : 21 : : 34 : 51, is a compound ratio. Or 7 × 22 : 11 × 21 : : 34 : 51, and applying Art. 40 we have 7 × 22 × 51 = 11 × 21 × 34.

45. Compound Proportion is also called the Double Rule of Three. It enables us to obtain the answer by a single statement, although two or more questions are contained in the question.

46. In Compound Proportion there are *three* or *more* ratios, one of which is imperfect and all the others perfect.

47. Let is be required to solve the following question: If 18 men dig a trench 30 yards long, in 24 days, by working 8 hours a day, how many men will dig a trench 60 yards long, in 64 days, working 6 hours a day?

Let us suppose the time to be the same in both cases, and this question becomes the same as the following

If 18 men dig 30 yards of trench, how many men will dig 60 yards?

Here it is evident the answer will be the same fraction of 18 that 60 yards is of 30 yards; or, in other words, the required number of men = $\frac{60}{30}$ of 18 men.

Next let us take into account the number of days; but suppose they work the same number of hours per day in both cases.

The question then becomes: If $\frac{60}{30}$ of 18 men require 24 days to dig a trench, how many men will dig it in 64 days?

In this case it is plain that the answer will be the same fraction of $\frac{60}{30}$ of 18 men that 24 days is of 64 days; that is, the required number of men = $\frac{24}{64}$ of $\frac{60}{30}$ of 18 men.

Lastly, let us take into consideration the time worked each day.

The question then becomes If $\frac{24}{64}$ of $\frac{60}{30}$ of 18 men dig a trench in a certain number of days, working 8 hours per day, how many men will dig it working 6 hours per day?

In this case the answer is obviously = $\frac{8}{6}$ of $\frac{24}{64}$ of $\frac{60}{30}$ of 18 men, or dividing these equals by 18, $\frac{Answer}{18} = \frac{8}{6} \times \frac{24}{64} \times \frac{60}{30}$.

Or taking the reciprocals $\frac{18}{Answer} = \frac{6}{8} \times \frac{64}{24} \times \frac{30}{60}$.

That is the ratio compounded of 6 : 8, 64 . 24, and 30 : 60 = ratio of 18 : Answer, or, $\left.\begin{matrix} 30 : 60 \\ 64 : 24 \\ 6 : 8 \end{matrix}\right\} :: 18 : \text{Answer}$.

The answer is equal to the continued product of the third term, and all the second terms, divided by the continued product of all the first terms.

From the preceding principles and illustrations, we deduce the following general

RULE FOR COMPOUND PROPORTION.

Place that number which is of the same kind as the answer in the third term, and the letter x to represent the answer in the fourth term.

Then take the other numbers in pairs, or two of a kind, and arrange them as in simple proportion.

Finally multiply together all the second terms and the third term, divide the result by the product of the first term, and the quotient will be the fourth term or answer required.

NOTE.—Since the third term and second terms multiplied together constitute a dividend, and the first terms multiplied together a divisor, we may (Arts. 79–84, Sect. II) cancel any factors that are common to any of the first terms and to the third term or any of the second terms.

EXAMPLE 1.—If 5 compositors, in 16 days, 11 hours long, compose 25 sheets of 24 pages in each sheet, 44 lines in each page and 40 letters in a line; in how many days, each 10 hours long, may 9 compositors compose a volume, to be printed in the same letter, consisting of 36 sheets, 16 pages to a sheet, 50 lines to a page, and 45 letters to a line?

STATEMENT. SAME CANCELLED.

$$\left.\begin{array}{rcl} 9 \text{ comp.} & : & 5 \text{ comp.} \\ 10 \text{ hours} & : & 11 \text{ hours.} \\ 25 \text{ sheets} & : & 36 \text{ sheets.} \\ 24 \text{ pages} & : & 16 \text{ pages.} \\ 44 \text{ lines} & : & 50 \text{ lines.} \\ 40 \text{ letters} & : & 45 \text{ letters.} \end{array}\right\} \begin{array}{c} \text{days.} \\ ::16:x \end{array} \qquad \left.\begin{array}{rcl} \cancel{9} & : & \cancel{5} \\ \cancel{10} & : & \cancel{11} \\ {}^{\cancel{5}}\cancel{25} & : & \cancel{36}^{\cancel{4}} \\ {}^{\cancel{3}}\cancel{24} & : & \cancel{16}^{\cancel{2}} \\ {}^{\cancel{4}}{}^{\cancel{20}}{}^{\cancel{11}}\cancel{44} & : & \cancel{50}^{\cancel{10}} \\ \cancel{40} & : & \cancel{45}^{\cancel{15}\,3} \end{array}\right\} \begin{array}{c} 4 \\ ::\cancel{16}:x. \\ \textit{Ans. } 3 \times 4 = 12 \\ \text{days.} \end{array}$$

EXPLANATION.—The imperfect ratio is that of 16 days to an unknown number of days. We place this ratio to the right-hand side, as in Simple Proportion. Now we compare each pair of terms with this ratio, in order to decide whether they constitute a ratio of greater or less inequality. Thus, if 5 compositors require 16 days, will 9 compositors require more or less? Evidently less; therefore it is a ratio of greater inequality, and we must write it 9 : 5. Next, if 11 hours to the day require 16 days, will 10 hours to the day require more or less?—more; therefore we must write 10 : 11. Next, if 25 sheets require 16 days, will 36 sheets require more or less?—more; therefore we write 25 : 36. Next, if 44 lines to a page require 16 days, will 50 lines to a page require more or less?—more; therefore we write 44 : 50. Lastly, if 40 letters to a line require 16 days, will 45 letters to a line require more or less?—more; therefore we write 40 : 45.

The statement is now complete, and we cancel as follows: 5 cancels 5, the first consequent, and reduces 25, the third antecedent, to 5, and 5 cancels this 5, and reduces 50, the fifth consequent, to 10, and 10 cancels this 10 and 10, the second antecedent. Again, 9 cancels the first antecedent and reduces 36, the third consequent, to 4, and 4 cancels this 4 and reduces 44, the fifth antecedent, to 11, and 11 cancels this 11 and 11. the second consequent. Again, 8 reduces 24 to 3 and 16 to 2, 3 cancels this 3 and reduces 45 to 15. 2 cancels the 2 resulting from the 16 and reduces 40 to 20, and 5 reduces this 20 to 4 and the 15 resulting from 45 to 3. Lastly, 4 cancels this 4 and reduces 16, the third term, to 4. There remain but 3 and 4 which multiplied together make 12. *Ans.*

EXAMPLE 2.—If 24 men can saw 90 cords of wood in 6 days when the days are 9 hours long, how many cords can 8 men saw in 36 days, when they are 12 hours long?

STATEMENT. SAME CANCELLED.

$$\left.\begin{array}{rcl} 24 \text{ men} & : & 8 \text{ men.} \\ 6 \text{ days} & : & 36 \text{ days.} \\ 9 \text{ hours} & : & 12 \text{ hours.} \end{array}\right\} \begin{array}{c} \text{cords.} \\ ::90:x. \end{array} \qquad \left.\begin{array}{rcl} {}^{4}\cancel{24} & : & \cancel{8}^{2} \\ & & \cancel{6} \\ \cancel{6} & : & \cancel{36} \\ \cancel{9} & : & 12 \end{array}\right\} \begin{array}{c} 10 \\ ::\cancel{90}:x. \\ \textit{Ans. } 10 \times 2 \times 12 = \\ 240 \text{ cords.} \end{array}$$

Here the imperfect ratio is 90 : *Ans.* If 24 men saw 90 cords, will 8 men saw more or less?—less; therefore it is a ratio of greater inequality, and we write 24 : 8. Next, if 6 days saw 90 cords of wood, will 36 days saw more or less?—more; therefore it is a ratio of less inequality, and we write 6 : 36 Lastly, if 9 hours per day saw 90 cords, will 12 hours per day saw more or less?—more; therefore it is a ratio of less inequality, and we write 9 : 12.

EXAMPLE 8.—If 248 men, in 5½ days, of 11 hours each, dig a trench of 7 degrees of hardness, 232½ yards long, 3⅔ wide, and 2⅓ deep; in how many days, of 9 hours long, will 24 men dig a trench of 4 degrees of hardness, 337½ yards long, 5⅗ wide, and 3½ deep?

STATEMENT.

$$\left.\begin{array}{rl} 24 & : 248 \text{ men.} \\ 9 & : 11 \text{ hours.} \\ 7 & : 4 \text{ degrees.} \\ 232\frac{1}{2} & : 337\frac{1}{2} \text{ yds. long.} \\ 3\frac{2}{3} & : 5\frac{3}{5} \text{ yds. wide.} \\ 2\frac{1}{3} & : 3\frac{1}{2} \text{ yds deep.} \end{array}\right\} :: 5\tfrac{1}{2} \text{ days} : \textit{Ans.} \text{ or, } \left\{\begin{array}{c} \frac{24}{1} : \frac{248}{1} \\ \frac{9}{1} : \frac{11}{1} \\ \frac{7}{1} : \frac{4}{1} \\ \frac{465}{2} : \frac{675}{2} \\ \frac{11}{3} : \frac{28}{5} \\ \frac{7}{3} : \frac{7}{2} \end{array}\right\} :: \frac{11}{2} : x.$$

The answer will be $\left(\frac{248}{1} \times \frac{11}{1} \times \frac{4}{1} \times \frac{675}{2} \times \frac{28}{5} \times \frac{7}{2} \times \frac{11}{2}\right) \div \left(\frac{24}{1} \times \frac{9}{1} \times \frac{7}{1} \times \frac{465}{2} \times \frac{11}{3} \times \frac{7}{3}\right)$

$= \frac{248}{1} \times \frac{11}{1} \times \frac{4}{1} \times \frac{675}{2} \times \frac{28}{5} \times \frac{7}{2} \times \frac{11}{2} \times \frac{1}{24} \times \frac{1}{9} \times \frac{1}{7} \times \frac{2}{465} \times \frac{3}{11} \times \frac{3}{7}.$

CANCELLED.

$$\frac{\overset{\cancel{8}}{\cancel{248}}}{1} \times \frac{\cancel{11}}{1} \times \frac{\overset{\cancel{2}}{\cancel{4}}}{1} \times \frac{\overset{\overset{\cancel{15}}{\cancel{75}}}{\cancel{675}}}{\cancel{2}} \times \frac{\overset{4}{\cancel{28}}}{\cancel{5}} \times \frac{\cancel{7}}{\cancel{2}} \times \frac{11}{\cancel{2}} \times \frac{1}{\underset{\cancel{3}}{\cancel{24}}} \times \frac{1}{\cancel{9}} \times \frac{1}{\cancel{7}} \times \frac{\cancel{2}}{\underset{\cancel{31}}{\cancel{465}}} \times \frac{\cancel{3}}{\cancel{11}} \times \frac{3}{\cancel{7}}$$

$= 4 \times 3 \times 11 = 132$ days.

EXERCISE 87.

1. If 120 bushels of corn last 14 horses 56 days, how many days will 90 bushels last 6 horses? *Ans.* 98 days.
2. If a wall of 28 feet high were built in 15 days by 63 men, how many men would build a wall 32 feet high in 8 days? *Ans.* 135 men.
3. If 1 lb. of thread make 3 yards of linen of 1¼ yards wide, how many pounds of thread would be required to make a piece of linen of 45 yards long and 1 yard wide? *Ans.* 12lb.
4. If 3 lb. of worsted make 10 yards of stuff of 1¼ yards broad, how many pounds would make a piece 100 yards long and 1¼ broad? *Ans.* 25lb.
5. If 12 horses in 5 days draw 44 tons of stones, how many horses would draw 132 tons the same distance in 18 days? *Ans.* 10 horses.
6. If 27s. are the wages of 4 men for 7 days, what will be the wages of 14 men for 10 days? *Ans.* £6 15s.
7. 3 masters, who have each 8 apprentices, earn $144 in 5 weeks—each consisting of 6 working days. How much would 5 masters, each having 10 apprentices, earn in 8 weeks, working 5½ days per week—the wages being in both cases the same? *Ans.* $440.

8. If 6 shoemakers, in 4 weeks, make 36 pair of men's and 24 pair of women's shoes, how many pair of each kind would 18 shoemakers make in 5 weeks?

Ans. 135 pair of men's and 90 pair of women's shoes.

9. A wall is to be built of the height of 27 feet; and 9 feet high of it are built by 12 men in 6 days. How many men must be employed to finish the remainder in 4 days? *Ans.* 36.

10. If a footman travels 130 miles in 3 days, when the days are 14 hours long, in how many days of 7 hours each will he travel 390 miles? *Ans.* 18.

11. If the price of 10 oz. of bread, when the flour is 1s. 10½d. per stone, is 1d., what must be paid for 3lb. 12 oz. when the flour is 2s. 6d. per stone? *Ans.* 8d.

12. If 5 compositors in 16 days of 14 hours long, can compose 20 sheets of 24 pages in each sheet, 50 lines in a page, and 40 letters in a line, in how many days of 7 hours long may 10 compositors compose a volume to be printed in the same letter, containing 40 sheets, 16 pages in a sheet, 60 lines in a page, and 50 letters in a line? *Ans.* 32 days.

13. If 336 men, in 5 days of ten hours each, dig a trench of 5 degrees of hardness, 70 yards long, 3 wide, and 2 deep, what length of trench of 6 degrees of hardness, 5 yards wide, and 3 deep, may be dug by 240 men in 9 days of 12 hours each?

Ans. 36 yards.

14. If a pasture of 16 acres will feed 6 horses for 4 months, how many acres will feed 12 horses for 9 months? *Ans.* 72 acres.

15. If 25 persons consume 300 bushels of corn in one year, how much will 139 persons consume in 7 years at the same rate?

Ans. 11676 bushels.

16. If 32 men build a wall 36 feet long, 8 feet high, and 4 feet wide, in 4 days, in what time will 48 men build a wall 864 feet long, 5 feet high, and 3 feet wide? *Ans.* 30 days.

17. If a regiment of 679 soldiers consume 702 bushels of wheat in 336 days, how many bushels will an army of 22407 soldiers consume in 112 days? *Ans.* 7722 bushels.

18. If 12 tailors in 27 days can finish 13 suits of clothes, how many tailors in 19 days of the same length can finish the clothes of a regiment of soldiers consisting of 494 men?

Ans. 648 tailors.

19. If 17 head of cattle consume 5 acres 2 roods 10 perches of pasture in 30 days, how many acres would be consumed by 40 head in 51 days? *Ans.* 22 acres 1 rood.

20. If 180 bricks, 8 inches long, and 2 wide, are required for a walk 20 feet long, and 6 feet wide, how many bricks will be required for a walk 100 feet long and 4 feet wide?

Ans. 600 bricks.

CONJOINED PROPORTION.

48. Conjoined Proportion is a kind of Compound Proportion, in which the ratio of one of the terms to its corresponding term is made to depend on equivalencies among the intermediate terms of the proportion.

49. Conjoined Proportion is sometimes called the Chain Rule from the peculiar manner in which the different pairs of terms are linked, as it were, together. It relates principally to exchanges between different countries, in respect to specie, weights, and measures, but is applicable to common business transactions.

50. EXAMPLE 1.—Suppose 7 yards of velvet in Toronto cost as much as 9 in Montreal, and 16 in Montreal as much as 24 in Paris, how many yards in Toronto will cost as much as 54 in Paris?

EXPLANATION.—This question may be stated as a problem in Compound Proportion as follows:

$$\left.\begin{array}{r} 9:16 \\ 24:54 \end{array}\right\} :: 7:x$$

The imperfect ratio is 7 yards Toronto to an unknown number of yards Toronto. Then, if 9 yards Montreal pay for 7 yards Toronto, will 16 yards pay for more or less?—more; therefore we write 9 : 16. Next, if 24 yards Paris pay for a certain number $\left(\frac{16\times7}{9}\right)$ yards Toronto, will 54 yards Paris pay for more or less?—more; therefore we write the ratio 24 : 54. Now (Art. 47) the answer $=\frac{16\times54\times7}{9\times24}$; and it is evident that we may consider all the factors of the numerator as antecedents, and all the factors of the denominator as consequents, and then make the statement thus:

STATEMENT.

$$\begin{array}{llcll} 7 \text{ yds.} & \text{Toronto} & = & 9 \text{ yds.} & \text{Montreal.} \\ 16 \text{ ``} & \text{Montreal} & = & 24 \text{ ``} & \text{Paris.} \\ 54 \text{ ``} & \text{Paris} & = & x \text{ ``} & \text{Toronto.} \end{array}$$

Since the left-hand numbers constitute a dividend and the right-hand numbers a divisor, we may cancel factors that are common. Merely writing the numbers and doing this we have—

SAME CANCELLED.

$$\begin{array}{rcl} 7 & = & \cancel{9} \\ 4\ \cancel{16} & = & \cancel{24}\,\cancel{4} \\ \cancel{6}\ \cancel{54} & = & x = 4 \times 7 = 28 \text{ yds. } \textit{Ans.} \end{array}$$

From the preceding principles and illustrations we deduce the following:

RULE FOR CONJOINED PROPORTION.

Write the equivalent terms, as they occur, right and left of the sign of equality, taking care that terms of the same name shall always be on opposite sides.

Multiply all the terms on the same side as the odd term for a dividend and all on the other side for a divisor. The quotient will be the required term.

EXAMPLE 2.—If 25 sheep eat as much hay as 19 goats, and 33 goats as much as 10 cows, and 38 cows as much as 22 horses, how many horses will eat as much as 60 sheep?

STATEMENT.

25 sheep = 19 goats
33 goats = 10 cows
38 cows = 22 horses
x horses = 60 sheep

Or writing the numbers merely, cancelling and applying the rule.

SAME CANCELLED.

$$\left\{\begin{array}{rrcll} \cancel{5} & & & & \\ \cancel{3} & \cancel{25} &=& \cancel{19} & \\ \cancel{2} & \cancel{33} &=& \cancel{10}^{2} & \\ & \cancel{38} &=& \cancel{22}^{\cancel{11}} & 4 \\ & x &=& \cancel{60}^{\cancel{20}} & \end{array}\right.$$

Ans. 4 × 2 = 8 horses.

Here, since the term 25 sheep is on the left-hand side, we put the odd term, 60 sheep, on the right-hand side.

NOTE.—The sign = in such questions, merely means equal in *value*, or equal *in time*, or equal *in effect*, &c.

EXAMPLE 3.—If 19 lbs. of tea in Guelph cost as much as 20 lbs. in Hamilton, and 7 in Hamilton as much as $9\frac{1}{2}$ lbs. in Quebec, and 30 lbs. in Quebec as much as $29\frac{3}{4}$ lbs. in Boston, and $8\frac{1}{2}$ lbs. in Boston as much as $5\frac{1}{3}$ lbs. in London, and 10 lbs. in London as much as 57 lbs. in Hong Kong; how many lbs. in Hong Kong are worth 100 lbs. in Guelph?

STATEMENT.

19 Guelph = 20 Hamilton
7 Hamilton = $9\frac{1}{2}$ Quebec
30 Quebec = $29\frac{3}{4}$ Boston
$8\frac{1}{2}$ Boston = $5\frac{1}{3}$ London
10 London = 57 Hong Kong
x Hong Kong = 100 Guelph

SAME CANCELLED.

$$\begin{array}{rrcll} & & & 10 & \\ & \cancel{19} &=& \cancel{20} & \\ & \cancel{7} &=& 9\frac{1}{2} & \\ \cancel{3} & \cancel{30} &=& \cancel{29\frac{3}{4}} & \cancel{4}\frac{1}{4} \\ \cancel{2} & \cancel{8\frac{1}{2}} &=& 5\frac{1}{3} & \\ & \cancel{10} &=& \cancel{57} & \cancel{19} \\ & x &=& \cancel{100} & \cancel{10} \end{array}$$

Ans. $10 \times 9\frac{1}{2} \times 5\frac{1}{3} = 506\frac{2}{3}$ lbs.

EXERCISE 88.

1. If 17 cords of wood are equivalent to 116 lbs. of tea, and 87 lbs. of tea to 23 barrels of flour, and 19 barrels of flour to 34 days' work, and 92 days' work to 57 baskets of peaches, and 31 baskets of peaches to 24 dollars, and 12 dollars to 2 tons of coal; how many cords of wood may be purchased for 35 tons of coal?
Ans. $135\frac{5}{8}$.

2. If 6 lbs. of tea are worth 29 lbs. of sugar, and 17 lbs. of sugar pay for 1 bushel of wheat, and 27 bushels of wheat are equivalent to 4 tons of coal, and 34 tons of coal purchase 15 cows, and 29

cows cost \$1160; how many pounds of tea can be purchased for \$20? *Ans.* 26$\frac{84}{290}$.

3. If 11 bushels of barley pay for 21 bushels of potatoes, and 19 bushels of potatoes for 29 bushels of oats, and 115 bushels of oats for 44 bushels of wheat, and 14½ bushels of wheat for 88 bushels of peas, and 60 bushels of peas for 55 bushels of rye, and 75 bushels of rye for 11½ bushels of clover seed; for how many bushels of barley will 36 bushels of clover seed pay? *Ans.* 87$\frac{4}{7}$.

4. If 16 baskets of pears pay for 29 turkeys, and 17 turkeys for 7 days' work, and 7¼ days' work for 187 loaves of bread, and 3½ loaves of bread cost as much as 4 lbs. of veal, and veal is 11 cents per pound, and \$7·92 pay for 63 lbs. of sugar; how many pounds of sugar will 21 baskets of pears purchase? *Ans.* 404¼.

5. Suppose A can do as much work in 7 days as B can in 11 days, and B as much in 5 days as C can in 8 days, and C as much in 15 days as D can in 21 days, and D as much in 11 days as E can in 5 days; in how many days would A do as much work as E can do in 42 days? *Ans.* 26¼.

6. If 7 barrels of flour pay for 23 cords of wood, and 6 cords of wood pay for 11 cwt. of beef, and 46 cwt. of beef cost £28, and £77 pay for 9 sheep, and 5 sheep are worth as much as 8 tons of coal; how many barrels of flour may be purchased for 9 tons of coal? *Ans.* 18$\frac{1}{8}$.

7. If 15s. in N. England be the same in value as 20s. in N. York, and 24s. in N. York the same as 22s. 6d. in N. Jersey, and 30s. in N. Jersey the same as 20s. in Canada; how many pounds in N. England are the same in value as £240 7s. 6d. in Canada? *Ans.* £288 9s.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers following the questions refer to the numbered articles of the section.*

1. In how many ways may one number be compared with another with respect to magnitude? (1)
2. What is ratio? (2)
3. What is the difference between the Geometrical and the Arithmetical ratio of numbers? (3)
4. How many ways have we of expressing the ratio of one number to another? (4)
5. Between what kind of quantities only can ratio exist? (5)
6. When are quantities said to be of the same kind? (6)
7. What is a couplet? (7)
8. What is the antecedent?—the consequent? (8)
9. How many kinds of ratio are there? (9)
10. What is a direct ratio? (10)
11. What is an inverse ratio? (11)
12. What is the reciprocal of a quantity? (13)
13. What is a reciprocal ratio? (12)
14. How is the reciprocal ratio of two numbers expressed? (14)

15. Show that "reciprocal ratio" and "inverse ratio" are interchangeable terms. (12)
16. What is a simple ratio? (15)
17. What is a compound ratio? (16)
18. Since a compound ratio does not differ in nature from a simple ratio, why is the term used? (17)
19. How are ratios compounded together? (18)
20. How does multiplying the antecedent or dividing the consequent of a couplet by any number, affect the ratio? (19)
21. How does dividing the antecedent or multiplying the consequent of a couplet by any number, affect the ratio? Why? (19)
22. How does multiplying or dividing both antecedent and consequent of a couplet by any number, affect the ratio? Why? (19)
23. How does it happen that we may cancel any factors common to an antecedent and a consequent, before compounding ratios together? (20)
24. When is a ratio called a *ratio of equality*? (21)
25. When is a ratio called a *ratio of greater inequality*? (21)
26. When is a ratio called a *ratio of less inequality*? (21)
27. How are ratios compared with one another? (22)
28. When equal ratios are added together, what is the nature of the resulting ratio? (23)
29. What effect has adding the same number to both terms of a ratio? (25 and 26)
30. What is Proportion? (27)
31. What are the terms of the two equal ratios called? (28)
32. How many ways are there of expressing Proportion? (29)
33. What is the supposed derivation of the sign ∷? (29—Note)
34. How many *terms* must there be in every proportion? (30)
35. When three *numbers* constitute a proportion, what is the repeated term called? What is the last term called? (31)
36. Point out the distinctions between ratio and proportion. (32)
37. What are "*extremes*" and "*means*"? (33)
38. Prove that if four quantities are proportional, the product of the extremes is equal to the product of the means. (34)
39. What is the test of geometrical ratio? (35)
40. Deduce from this principle a rule for finding any one of the terms when the other three are given. (36)
41. If $r : w :: x : y$, what does the proportion become? 1st, by composition; 2nd, alternately; 3rd, by conversion; 4th, by division; 5th, inversely. (37)
42. What are the different kinds of Proportion? (38)
43. What other names has Simple Proportion? Why so called? (39)
44. Give the rule for making the statement in Simple Proportion. (40)
45. Give the rule for finding the unknown quantity after the statement is made. (40)
46. Show that we may cancel any factors that are common to the first term and either of the others, before applying the rule. (41)
47. If any of the terms contain fractions, what is done? (42)
48. If the first and second terms are not of the same denomination, what is the rule? (43)
49. What is Compound Proportion? (44)
50. What other name has Compound Proportion? (45)
51. How many ratios are there in Compound Proportion, and how many of them are perfect? (46)
52. In stating a question in Compound Proportion, what do you make the third term? (47)
53. How do you know whether the other ratios are ratios of greater or less inequality? (47)
54. When the statement is made, how is the answer obtained? (47)
55. Show that before applying the rule we may cancel any factors, that are common to any of the first terms, and to the second and third terms. (47—Note)

56. What is Conjoined Proportion? (48)
57. Why is it sometimes called the Chain Rule? (49)
58. Give the rule for Conjoined Proportion. (50)
59. In what sense is the sign = taken in these statements? (50)

EXERCISE 89.

MISCELLANEOUS EXERCISE.

(On preceding Rules.)

1. What is the ratio compounded of the ratios 7 : 8, 17 : 11, 23 : 29, 319 : 119, and 16 : 69?
2. Reduce £119 16s. 6½d. to dollars and cents.
3. How many days are there from 12th March to the 17th of the following February?
4. Compare together the following ratios, and point out which is greatest and which least, 9 : 13, 21 : 27, 7 : 10, and 11 : 15.
5. From $76{\cdot}23\dot{4}7\dot{8}$ take $19{\cdot}134\dot{2}29\dot{1}$.
6. Multiply 71324*t* *undenary* by 23421 *quinary* and divide the result by *t4e7 duodenary*. Give the answer in each scale.
7. If 5·63 cubic inches of water weigh 3·254 ounces avoirdupois, what will be the weight of 7·9 cubic inches of nitric acid having a specific gravity of 1·220?
8. Divide 63 yds. 3 qrs. 2 na. 1 in. of ribbon equally among 17 persons.
9. What is the value of ·913625 of an acre at 67 cents per sq. yard?
10. Multiply $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{7}{8}$ of 20 bushels by $\cdot 5 \times \cdot\dot{6} \times \frac{7}{8}$.
11. Of the ratios 6 : 7, 17 : 8, 23 : 11, and 88 : 176, point out (1) which is the greatest, (2) which is the least, (3) which are ratios of greater inequality, (4) which are ratios of less inequality, (5) what is the ratio compounded of these ratios.
12. The population in Canada in 1851 was 1842265, and in 1857 it was estimated at 2571437. What was the rate per cent. of increase?
13. From one-half of two-thirds of eighteen twenty-ninths subtract one-eighth of two-thirds of five-sevenths.
14. Deduct 7 per cent. from 11 feet.
15. What is the value of 79 lbs. of tea at £·00163 per ounce?
16. If 3 men in 2½ days, working 12 hours a day, can cradle a field of wheat containing 20 acres, in how many days can 4 men, working 10 hours a day, cradle a field of wheat containing 35 acres?
17. Find the value of $(\frac{4}{5} \text{ of } \frac{9}{11} \times \cdot 02 \times \cdot 4\dot{5}\dot{6}) \div (\frac{16}{17} \text{ of } \frac{2}{3} \text{ of } \frac{1}{8} \text{ of } 51)$.
18. A certain number is divided by 5, the result is divided by $\frac{7}{2}$, this result by $\frac{13}{5}$, and this last result by $\frac{4}{7}$. The last quotient is 2; what was the original number?

19. If 50 barrels of flour in Toronto are worth 125 yards of cloth in New York, and 80 yards of cloth in New York 6 bales of cotton in Charleston, and 13 bales of cotton in Charleston $8\frac{1}{2}$ hogsheads of sugar in New Orleans; how many hogsheads of sugar in New Orleans are worth 1000 barrels of flour in Toronto?
20. Multiply 73·47 by ·0063, and divide the result by $17\cdot2\dot{3}4\dot{5}$.
21. Reduce 2 roods 7 per. 4 yds. 3 ft. 117 in. to the decimal of 7 acres.
22. Deduct ·73 of 11 furlongs from $\frac{3}{7}$ of $\frac{4}{5}$ of $\frac{1}{3}$ of 70 miles.
23. From 274312 *nonary* take 1101011010 *binary*, and multiply the result by 5555 *septenary*. Give the answer in all three scales.
24. Find the l. c. m. of 44, 275, 18, 190, 209, and 225.
25. If 60 men in 6 weeks of 5 working days, of 10 hours each, build an embankment 800 yards in length, 18 feet in mean breadth and 11 ft. in mean height, how many men will make an embankment 8742 feet long, 20 feet wide and 8 ft. high, in 10 weeks, of 6 days each, and eleven working hours to each day?
26. How many divisors has the number 172000?
27. Multiply $42\cdot\dot{7}$ by $9\cdot7\dot{1}2\dot{3}$.
28. Deduct 27 per cent. from $73·42.
29. What are all the divisors of 6300?
30. If $\frac{3}{7}$ of $\frac{3}{5}$ of $3\frac{1}{2}$ lbs. of coffee cost $\frac{2}{7}$ of $\frac{3}{5}$ of $\frac{20}{27}$ of $\frac{1}{2}$ of a dollar, what will $\frac{3}{5}$ of $\cdot\dot{7}$ of ·6 of $\frac{21}{50}$ of 90 lbs. cost?
31. If $2739·18 be divided among 7 men, 2 women, and 11 children, so that each child shall have $\frac{2}{7}$ of a woman's share, and each woman $\frac{3}{11}$ of a man's share, what will be the amount received by each?
32. What is the reciprocal ratio of $\frac{2}{7}$: $\frac{11}{3}$; the direct ratio of 93 : 17, and the inverse ratio of $\frac{2}{5}$ of $\frac{7}{8}$?
33. Add together $\frac{4}{9}$ of $6\frac{1}{2}$ yards, $\frac{3}{5}$ of $\frac{4}{7}$ of $8\frac{2}{3}$ ft., and $\frac{2}{7}$ of $\frac{3}{11}$ of $7\frac{7}{10}$ inches.
34. What is the ratio compounded of 23 : 7, 4 : 11, 6 : 5, 13 : $11\frac{1}{3}$, and $38\frac{1}{3}$: 3?
35. A pint contains 9000 grains of barley, and each grain is one third of an inch long. How far would the grains in 23 bush. 2 pks. 1 gal. 1 qt. 1 pt. reach if placed one after another?
36. Reduce $\frac{4158}{10395}$ to its lowest terms.
37. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{2}{7}$ in the *octenary* scale.
38. If 17 sheep eat as much grass as 6 cows, and 26 cows require $27\frac{1}{3}$ acres, and 12 acres supply 13 horses, and 11 horses eat as much as 28 goats, how many goats will eat as much as 68 sheep?
39. Suppose that 50 men, by working 5 hours each day, can dig, in 54 days, 24 cellars, which are each 36 feet long, 21 feet wide, and 10 feet deep, how many men would be required to dig, in 27 days, 18 cellars, which are each 48 feet long, 28 feet wide, and 9 feet deep, provided they work only 3 hours each day?

SECTION VI.

PRACTICE.

1. Practice is so called from its being the method of calculation *practised* by mercantile men; it is an abridged mode of performing processes dependent on the Rule of Three—particularly when one of the terms is unity.

The *statement* of a question in practice, in *general terms*, would be—*One quantity of goods : another quantity of goods : : price of former : price of latter.*

2. The simplification of the Rule of Three by means of *practice*, is principally effected, either by dividing the given *quantity* into "parts," and finding the sum of the prices of these parts; or by dividing the *price* into "parts," and finding the sum of the prices of each of these parts; in either case, as is evident, we obtain the required price.

3. An Aliquot Part is an exact or even part.

Thus, 2 shillings is an aliquot part of a pound; 12½ cents is an aliquot part of a dollar; 6 months, 4 months, 3 months, 2 months, 1½ months are aliquot parts of a year, &c.

TABLE OF ALIQUOT PARTS.

Parts of $1.	Parts of a year.	Parts of a month.	Parts of £1.	Parts of 1s.	Parts of a cwt.* of 112 lbs.
50 cts. = ½	6 m'ths = ½	15 days = ½	10s = ½	6d = ½	56 lb = ½
33⅓ = ⅓	4 = ⅓	10 = ⅓	6s 8d = ⅓	4d = ⅓	28 lb = ¼
25 = ¼	3 = ¼	7½ = ¼	5s = ¼	3d = ¼	16 lb = 1/7
20 = ⅕	2 = ⅙	6 = ⅕	4s = ⅕	2d = ⅙	14 lb = ⅛
16⅔ = ⅙	1½ = ⅛	5 = ⅙	3s 4d = ⅙	1½ = ⅛	8 lb = 1/14
12½ = ⅛	1 = 1/12	3 = 1/10	2s 6d = ⅛	1d = 1/12	7 lb = 1/16
10 = 1/10		2 = 1/15	2s = 1/10		parts of a qr. of 28 lbs.
8⅓ = 1/12		1 = 1/30	1s 8d = 1/12		14 lb = ½
6¼ = 1/16			1s 4d = 1/15		7 lb = ¼
5 = 1/20			1s 3d = 1/16		3½ lb = ⅛
4 = 1/25			1s = 1/20		1¾ lb = 1/16
2 = 1/50					

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* Although we allow but 100 lbs. to the cwt. in Canada, it is often necessary to make calculations with the old cwt. of 112 lbs. This arises from the

Example 1.—Find the price of 2783 yards of silk at $3·37½ per yard.

OPERATION.

25 c.	¼	2783
		3
		8349
12½ c.	½	695·75
		347·87½
		Ans. $9392·62½

The cost of 2783 yards at $3·37½=cost at $3+cost at 37½ cents.

2783 yds. at $3 comes to 3 times as much as at $1; i. e., to 3 times $2783, or $8349. 37½ cts. equals 25 cts.+12½ cents, hence, 2783 yds. at 37½ cents=price at 25 cents+price at 12½ cents.

Since 2783 yards at $1 come to $2783, and 2[illegible] ¼ of a dollar; 2783 yards at 25 cents come to ¼ of $2[illegible], to $695·75. Again, because 2783 yards at 25 cents [illegible] to $695·75 and 12½ cents equals ½ of 25 cents, 2783 yards at 12½ cents would [illegible]me to ½ of $695·75; i. e., to $347·87½.

Then 2783 yards at $3·37½=price at $3+price at 25 cents+price at 12½ cents =$8349+$695·75+$347·87½=$9392·62½.

Example 2.—What is the cost of 972 oz. of gold dust at £3 14s. 8¼d. per oz.?

OPERATION.

10s.	½	972						
		3						
		£2916			= cost at	£3	0	0
3s. 4d.	⅓	486			= cost at	0	10	0
10d.	¼	162			= cost at	0	3	4
5d.	½	40	10s.		= cost at	0	0	10
1¼d.	¼	20	5		= cost at	0	0	5
		5	1	3d.	= cost at	0	0	1¼
		£3629	16	3	= cost at	£3	14	8¼

Example 3.—Find the price of 729 days' work at £1 7s. 1½d. per day.

OPERATION.

5s.	¼	£729	0	0	= price at	£1	0	0
1s. 8d.	⅓	182	5	0	= price at	0	5	0
5d.	¼	60	15	0	= price at	0	1	8
½d.	$\frac{1}{10}$	15	3	9	= price at	0	0	5
			15	2½	= price at	0	0	0½
		£987	18	11½	= price at	£1	7	1½

Example 4.—What is the cost of 624 bush. 1 pk. 1 gal. 3 qt. of wheat at $2·87½ per bushel?

OPERATION.

50 cts.	¼	624	
		2	
		$1248	= price of 624 bush. at $2·00
25 cts.	½	812	= price " " at 50
12½ cts.	½	156	= price " " at 25
		78	= price " " at 12½
		$1794	= price of 624 bush. at $2·87½

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fact that the latter is still in common use in Great Britain, several of the States of the American Union, &c. The aliquot parts of the new cwt. of 100 lbs. are the same as the aliquot parts of $1.

1 pk.	$\frac{1}{4}$	\$2·87$\frac{1}{2}$	= price of 1 bush.
1 gal.	$\frac{1}{2}$	·71$\frac{7}{8}$	= price of 1 pk.
2 qt.	$\frac{1}{2}$	·35$\frac{15}{16}$	= price of 1 gal.
1 qt.	$\frac{1}{2}$	·17$\frac{31}{32}$	= price of 2 qt.
		·08$\frac{63}{64}$	= price of 1 qt.
		\$1·34$\frac{49}{64}$	= price of 1 pk. 1 gal. 3 qt.

Then \$1794 = price of 624 bushels at \$2·87$\frac{1}{2}$ per bushel,
1·34$\frac{49}{64}$ = price of 1 pk. 1 gal. 3 qt. at \$2·87$\frac{1}{2}$ per bush.

\$1795·34$\frac{49}{64}$ = price of 624 bush. 1 pk. 1 gal. 3 qt. at \$2·87$\frac{1}{2}$ per bush.

EXAMPLE 5.—What is the price of 96 acres 1 rood 14$\frac{1}{2}$ per. at £7 11s. 5$\frac{1}{4}$d. per acre?

10s.	$\frac{1}{2}$	96		
		7		
		£672	0	= price of 96 acres at £7 0 0
1s. 3d.	$\frac{1}{8}$	48	0	= " " " at 0 10 0
1$\frac{1}{2}$d.	$\frac{1}{10}$	6	0	= " " " at 0 1 3
$\frac{3}{4}$d.	$\frac{1}{2}$		12	= " " " at 0 0 1$\frac{1}{2}$
			6	= " " " at 0 0 0$\frac{3}{4}$
		£726	18	= price of 96 acres at £7 11 5$\frac{1}{4}$

1 rood	$\frac{1}{4}$	£7 11 5$\frac{1}{4}$	
		1 17 10$\frac{1}{4}$+$\frac{1}{4}$	= price of 1 rood.
10 per.	$\frac{1}{4}$	9 5$\frac{1}{2}$+$\frac{5}{16}$	= price of 10 perches.
4 per.	$\frac{1}{10}$	3 9$\frac{1}{4}$+$\frac{29}{40}$	= price of 4 perches.
$\frac{1}{2}$ per.	$\frac{1}{8}$	5$\frac{1}{2}$+$\frac{229}{320}$	= price of $\frac{1}{2}$ perch.

£2 11 7 + $\frac{1}{320}$ f. = price of 1 rd. 14$\frac{1}{2}$ per. at £7 11s. 5$\frac{1}{4}$d. per ac.
£726 18 = price of 96 acres.

Ans. £729 9s. 7d. + $\frac{1}{320}$ f. = price of 96 acres 1 rood 14$\frac{1}{2}$ per.

EXAMPLE 6.—What is the cost of 964$\frac{11}{15}$ square yards of plastering at 22$\frac{1}{2}$ cents per square yard?

20 cts.	$\frac{1}{5}$	964	
2$\frac{1}{2}$ cts.	$\frac{1}{8}$	\$192·80	= cost of 964 yds. at 20 cts.
		24·10	= cost of 964 yds. at 2$\frac{1}{2}$ cts.
		\$216·90	= cost of 964 yds. at 22$\frac{1}{2}$ cts.
		·16$\frac{1}{2}$	= cost of $\frac{11}{15}$ of a yd. at 22$\frac{1}{2}$ cts.

$$\frac{22\frac{1}{2}\times 11}{15} = 16\tfrac{1}{2} \text{ cents.}$$

Ans. \$217·06$\frac{1}{2}$ = cost of 964$\frac{11}{15}$ yds. at 22$\frac{1}{2}$ cts. per yd.

EXERCISE 90.

1. Required the value of 92647 lbs. of tea at 35 cents per lb. *Ans.* $32426·45.
2. What is the cost of 94987 pails at 1s. 5d. each? *Ans.* £6723 14s. 1d.
3. What is the worth of 95972 boxes at 7½ cents? *Ans.* $7197·90.
4. What is the cost of 62 acres at $28·80 per acre? *Ans.* $1785·60.
5. Find the price of 2310 lbs. at 32½ cents per lb. *Ans.* $750·75.
6. Find the price of 2117 bags at 37½ cents each. *Ans.* $793·87½.
7. Find the price of 7506 pair of shoes at 1s. 9¾d. a pair. *Ans.* £680 4s. 7½d.
8. What is the value of 1217 lbs. of coffee at 17½ cents per lb? *Ans.* $212·97½.
9. Find the price of 2103 cords of wood at $3·07½ per cord. *Ans.* $6466·72½.
10. What is the cost of 2096 oz. of gold dust at £3 18s. 10½d. per oz.? *Ans.* £8266 2s. 0d.
11. Required the value of 6 oz. 18 dwt. 20 grs. of silver at $1·55 per oz. *Ans.* 10·75$\frac{23}{24}$.
12. What is the cost of 98 yds. 3 qrs. 1 na. of cloth at £1 15s. per yard? *Ans.* £172 18s. 5¼d.
13. What is the rent of 344 acres 3 roods 15 per. at £4 1s. 1d. per acre? *Ans.* £1398 1s. 0$\frac{31}{32}$d.
14. What is the price of 5 oz. 6 dwt. 17 grs. of mercury at 5s. 10d. per oz.? *Ans.* £1 11s. 1$\frac{23}{48}$d.
15. Find the price of 4 yards 2 qrs. 3 nails of satin at £1 2s. 4d. per yard. *Ans.* £5 4s. 8¼d.
16. Find the price of 32 acres 1 rood 14 perches at £1 16s. per acre. *Ans.* £58 4s. 1⅘d.
17. Find the price of 3 gals. 5 pts. of spirits of wine at 7s. 6d. per gallon. *Ans.* £1 7s. 2¼d.
18. How much will 724 bushels of apples come to at $1·67½ per bushel? *Ans.* $1212·70.
19. What is the cost of 721 bush. of wheat at $1·93¾ per bush.? *Ans.* $1396·93¾.
20. What is the cost of 4514 rods of fencing at £2 17s. 7½d. per rod? *Ans.* £13005 19s. 3d.
21. What is the price of 3749⅞ acres at £3 15s. 6d. per acre? *Ans.* £14155 17s. 9¾d.

Allowing 112 lbs to the cwt., find the value of—

22. 17 cwt. 1 qr. 17 lbs. at £1 4s. 9d. per cwt. *Ans.* £21 10s. 8$\frac{37}{112}$d.
23. 78 cwt. 3 qrs. 12 lbs. at $11·55 per cwt. *Ans.* $910·80.
24. 20 tons 19 cwt. 3 qrs. 27½ lbs. at £10 10s. per ton. *Ans.* £220 9s. 11½d. nearly.
25. 219 tons 16 cwt. 3 qrs. at $45·50 per ton. *Ans.* $10002·60⅝.

EXERCISE 91.

BILLS OF PARCELS.

(No. 1.)

QUEBEC, 16*th April*, 1859.

Mr. JOHN DAY,

Bought of RICHARD JONES.

	s.	d.		£	s.	d.
15 yards of fine broadcloth, at..............	13	6	per yard,	10	2	6
24 yards of superfine ditto, at..............	18	9	"	22	10	0
27 yards of yard wide ditto, at..............	8	4	"	11	5	0
16 yards of drugget, at..............	6	3	"	5	0	0
12 yards of serge, at..............	2	10	"	1	14	0
32 yards of shalloon, at..............	1	8	"	2	13	4

Ans. £53 4 10

(No. 2.)

MONTREAL, 24*th June*, 1859.

Mr. JAMES PAUL,

Bought of THOMAS NORTON.

	s.	d.	
9 pair of worsted stockings, at..............	4	6	per pair,
6 pair of silk ditto, at..............	15	9	"
17 pair of thread ditto, at..............	5	4	"
23 pair of cotton ditto, at..............	4	10	"
14 pair of yarn ditto, at..............	2	4	"
18 pair of women's silk gloves, at..............	4	2	"
19 yards of flannel, at..............	1	7½	per yard,

Ans. £23 15 4½

(No. 3.)

TORONTO, 10*th July*, 1859.

Mr. WILLIAM FILBERT,

Bought of GEORGF PRICE.

75½ lbs. of sugar, at..............	7¾	cents per lb.,
63 lbs. of tea, at..............	93	"
126 lbs. of butter, at..............	13	"
85⅜ lbs. of raisins, at..............	18¾	"
17 lbs. of sago, at..............	15	"
28 lbs. of rice, at..............	9	"
56¼ lbs. of starch, at..............	22	"

Ans. $105·02¾.

(No. 4.)

Mr. JOHN JAMES,

HAMILTON, 12th *August*, 1859.

Bought of JAMES THOMAS.

	$ cts.
198 Sangster's National Arithmetic, at	0·60
197 Robertson's Philosophy of Grammar, at	0·50
83 Hodgins' Geography, at	1·00
57 Sangster's Algebraic Formula, at	0·12½
217 Strachan's Canadian Penmanship, at	0·37½
143 Hodgins' Geography of British Provinces, at	0·45
227 Sangster's Elementary Arithmetic, at	0·30

Ans. $521·25

(No. 5.)

Mr. ALEX. LEITH,

NIAGARA, 17th *September*, 1859.

Bought of LAWRENCE MERCER

	s.	d.	
9½ yards of silk, at	12	9	per yard,
13 yards of flowered ditto, at	15	6	"
11¾ yards of lustring, at	6	10	"
14 yards of brocade, at	11	8	"
12¼ yards of satin, at	10	8	"
11⅜ yards of velvet, at	18	0	"

Ans. £44 15 10

(No. 6.)

Dr. ALEX. HAMILTON,

KINGSTON, 11th *July*, 1859.

Bought of TIMOTHY PESTLE.

14 oz. ipecacuanha, at	$0·67
23 " laudanum, at	0·89
17 " emetic tartar, at	1·25
25 " cantharides, at	2·17
27 " gum mastic, at	0·61
56 " gum camphor, at	0·27

Ans. $186·94

(No. 7.)

London, C. W., 1*st* *May*, 1859.

Mr. Jas. Grey,

Bought of Michael Lewis.

	s.	d.	
15½ lbs. of currants, at	0	4	per lb.,
17¼ lbs. of Malaga raisins, at	0	5½	"
19¾ lbs. of sun raisins, at	0	6	"
17 lbs. of rice, at	0	3½	"
8½ lbs. of pepper, at	1	6	"
3 loaves of sugar, weight 32½ lbs., at	0	8½	"
13 oz. of cloves, at	0	9	per oz.

Ans. £3 13 5⅛

TARE AND TRET.

4. *Tare* and *Tret* is the name given to a rule by means of which merchants calculate the amount of certain allowances which were formerly made in buying and selling goods by weight in large quantities. They were as follows:

1. Tret, an allowance for waste in weighing.
2. Tare, an allowance for the actual or supposed weight of the *box*, *bag*, *barrel*, &c., containing the goods. And
3. Cloff, an allowance of 2 lbs. in every 336 for the turn of the scale in retailing goods.

Of these the only one known in Canada is Tare; and as this is always set down in full in the invoice, Tare and Tret, as a rule, has no existence in Canadian mercantile transactions, and has therefore been altogether omitted.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*The numbers after the questions refer to the articles of the section.*

1. What is Practice? (1)
2. Why is it so called? (1)
3. Of what rule is Practice merely a modification? (1)
4. What would be the general statement of a question in Practice? (1)
5. How is the process of finding the price of a number of articles simplified by Practice? (2)
6. What is an aliquot part? (3)
7. What are the aliquot parts of a dollar? (3)

8. What are the aliquot parts of a year? (3)
9. What are the aliquot parts of a month? (3)
10. What are the aliquot parts of a £? (3)
11. What are the aliquot parts of a shilling? (3)
12. What are the aliquot parts of a cwt. (112 lbs.)? (3)

EXERCISE 92.

MISCELLANEOUS EXERCISE.

(*On preceding Rules.*)

1. Take the number 70204, and by removing the decimal point (1) multiply it by 100000; (2) divide it by 10000; (3) make it *thousandths;* (4) make it *tenths of billionths;* (5) make it *tenths;* and (6) make it *hundredths of billionths.*
2. Divide 427·1 by ·0000637.
3. What will 19 tons 19 cwt. 3 qrs. 27½ lbs. of hops cost, at £19 19s. 11¾d. per ton?
4. Add together $78{\cdot}\dot{7}2\dot{3}$, $11{\cdot}34\dot{2}$, $16{\cdot}713$, $19{\cdot}\dot{0}3\dot{4}$, $718{\cdot}213\dot{4}3\dot{7}$, and $12{\cdot}\dot{3}4567\dot{8}$.
5. Of the ratios 5 : 7, 9 : 13, 12 : 17, and 7 : 10, point out (1) which is greatest, (2) which is least, (3) what is the ratio compounded of these?
6. If 1 acre of land cost $80·50, what will 25 acres, 2 roods, 35 rods cost?
7. What is the G. C. M. of 144, 485, and 63.
8. What is the price of 7439 cords of wood at $3·68¾ a cord?
9. Reduce $\frac{135795}{222210}$, $\frac{714285}{999999}$, $\frac{109375}{1000000}$, and $\frac{30301}{33633}$ to their lowest terms.
10. If 34½ bushels of turnips are worth 17 bushels of potatoes, and 9 bushels of potatoes 59½ lbs. of tea, and 6 lbs. of tea 11½ stone of flour, and 13 stone of flour $3·60, and 38 cents pay for 12 lbs. of bread; how many bushels of turnips are worth 119 lbs. of bread?
11. If 27 men in 7 days, working 8 hours a day, paint 42 floors, each 20 feet long and 16 feet wide, with three coats of paint to each; in how many days, of 11 hours each, will 54 men paint 77 floors, each 24 feet long and 22 feet wide, giving each 5 coats of paint?
12. Take the number 7449164 and by removing the decimal point, make it (1) One hundred thousand times greater.
    (2) One million times less.
    (3) Hundredths of quadrillionths.
    (4) Thousandths.
    (5) Tenths of billionths.
    (6) Tenths.

13. Reduce 72342 *nonary* to equivalent expressions in the *duodenary*, *senary*, and *ternary* scales, and prove the results by reducing all four numbers to the decimal scale.
14. Express in the decimal scale the greatest and least numbers that can be formed with six digits in the *binary*, *quaternary*, *senary*, *octenary*, and *duodenary* scales.
15. Write down all the divisors of 1728.
16. What is the l. c. m. of the first fifteen *even* numbers, 2, 4, 6, 8, &c.?
17. From $97{\cdot}91\dot{3}4\dot{2}$ take $18{\cdot}123\dot{4}56\dot{7}$.
18. What would be the cost of painting a ceiling 20 ft. 7 in. long and 19 ft. 5 in. 7″ wide, at $2 87½ per square yard?
19. Divide 916 acres, 3 roods, 17 per., 7 yards, by 48 acres, 1 rood, 2 per., 17 yds.

SECTION VII.

PERCENTAGE, COMMISSION, BROKERAGE, STOCKS, INSURANCE, CUSTOM-HOUSE BUSINESS, ASSESSMENT.

1. The term Per Cent. is derived from the Latin word *per*, "by" or "for" and *centum*, "a hundred," and means "for a hundred." The term is usually employed to indicate the allowance paid for the use of money, but may also be used to express so much the hundred units of any other quantity.

Thus, the term 5 per cent. on so many dollars, gallons, miles, days, &c., signifies $5 on every $100, or 5 gallons on every 100 gallons, or 5 miles on every 100 miles, or 5 days on every 100 days, &c.

2. When the rate per cent. is known, the rate per unit is easily obtained by dividing the rate per cent. by 100.

Thus, 1 per cent. is equal to $\frac{1}{100}$ or ·01 per unit.
2 per cent. is equal to $\frac{2}{100}$ or ·02 per unit.
7 per cent. is equal to $\frac{7}{100}$ or ·07 per unit.
9 per cent. is equal to $\frac{9}{100}$ or ·09 per unit.
10 per cent. is equal to $\frac{10}{100}$ or ·10 per unit.
18 per cent. is equal to $\frac{18}{100}$ or ·18 per unit.
89 per cent. is equal to $\frac{89}{100}$ or ·89 per unit.
95 per cent. is equal to $\frac{95}{100}$ or ·95 per unit.
125 per cent. is equal to $\frac{125}{100}$ or 1·25 per unit.
878 per cent. is equal to $\frac{878}{100}$ or 8·78 per unit.

½ per cent. is equal to $\frac{\frac{1}{2}}{100}$ or ·005 per unit.

¼ per cent. is equal to $\frac{\frac{1}{4}}{100}$ or ·0025 per unit.

¾ per cent. is equal to $\frac{\frac{3}{4}}{100}$ or ·0075 per unit.

⅛ per cent. is equal to $\frac{\frac{1}{8}}{100}$ or ·00125 per unit.

6½ per cent. is equal to $\frac{6\frac{1}{2}}{100}$ or ·065 per unit, &c.

EXERCISE 93.

1. What rate per unit is equivalent to 1·6 per cent., 11 per cent., 17 per cent., 63 per cent.?
2. What rate per unit is equivalent to 6 per cent., 25 per cent., 137 per cent.?
3. What rate per unit is equivalent to 8½ per cent., 9¼ per cent., 2¾ per cent.?
4. What rate per unit is equivalent to ⅕ per cent., ⅞ per cent., 8¾ per cent.?
5. At 6¼ per cent., how much is it for 1? *Ans.* ·0625.
6. At 18⅗ per cent., how much is it for 1? *Ans.* ·186.
7. At 23⅝ per cent., how much is it for 1? *Ans.* ·23625.
8. At 2·734 per cent., how much is it for 1? *Ans.* ·02734.
9. At 82·7 per cent., how much is it for 1? *Ans.* ·827.
10. At 19⅓ per cent., how much is it for 1? *Ans.* ·19$\dot{3}$.

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3. To find the percentage of any given number—

RULE.

Multiply the given number by the rate per unit expressed decimally, and point off the product as directed in Art. 58, Sec. II.

EXAMPLE 1.—What is 7 per cent. on $678·93?

OPERATION.

$678·93 × ·07=$47·1751.

EXPLANATION.—7 per cent. is equivalent to ·07 per unit; or, in other words, the percentage on each dollar is 7 cents. It is obvious then that the percentage on the whole sum will be as many times 7 cents as the sum contains dollars; that is ·07 × 678·93.

EXAMPLE 2.—What is 6½ per cent. on $2934?

Ans. $2934 × ·065=$190·71.

EXAMPLE 3.—What is 47¾ per cent. on 7893 gallons of molasses?

Ans. 7893 gal. × ·4775=3768·9075 gallons.

EXERCISE 94.

1. What is 5 per cent. of $742·10? *Ans.* $37·10½.
2. What is 11 per cent. of $1000? *Ans.* $110.
3. How much is 10 per cent. of $784·19? *Ans.* $78·419.

4. How much is 87½ per cent. of $1624·50? *Ans.* $1421·4875.
5. What is 12½ per cent. on $994·70? *Ans.* $124·3375.
6. What is 8¾ per cent. on $777·50? *Ans.* $68·03⅛.
7. What is 2¼ per cent. of $7135·80? *Ans.* $160·5555.
8. A merchant imports 2740 boxes of oranges, and finds, upon receiving them, that 20 per cent. of the whole quantity are decayed. To how many boxes was his loss equivalent? *Ans.* 548 boxes.
9. A gentleman purchases a farm for $7490, agreeing to pay 10 per cent. down, 17 per cent. at the end of the first year, 27 per cent. at the end of the second year, and 46 per cent. at the end of the third year. What is the amount of each payment?
   Ans. $749 down.
   $1273·30 at the end of 1st year.
   $2022·30 at the end of 2nd year.
   $3445·40 at the end of 3rd year.
10. What is the difference between 4½ per cent. of $740 and 2½ per cent. of $1680? *Ans.* $8·70.
11. If I purchase 729 gallons of brandy and lose 11 per cent. by leakage, &c., how much have I remaining? *Ans.* $648\frac{81}{100}$ gallons.
12. Add together 25 per cent. of $763·22, 16 per cent. of $847·16, and 6¼ per cent. of $1234·17. *Ans.* $403·486225.
13. A person dying leaves an estate worth $17429·40 to be divided among his three sons. The eldest is to receive 43 per cent. of the whole, the second 37 per cent. of the whole, and the youngest son the remainder; what is the share of each?
   Ans. The eldest receives $7494·64⅕, the second $6448·87⅘, and the youngest $3485·88.
14. A merchant purchases vinegar to the amount of 68978 gallons, and finds, upon receiving it, that 36 per cent. had leaked away. What was his loss? *Ans.* 24832·08 gallons.
15. A brick kiln contains 29800 bricks, and it is found after burning that 17 per cent. of the entire quantity are worthless; how many good bricks were there in the kiln? *Ans.* 24734.

COMMISSION.

4. Commission is the percentage charged by agents, or commission merchants, for their services in purchasing or selling goods, collecting bills, &c.

The person who buys or sells goods for another is called an Agent, a Commission Merchant, a Factor, or a Correspondent.

5. To find the commission of any sum at a given rate per cent. is simply to find the percentage on that sum, and the rule employed is the same as that in Art. 3, viz:

Multiply the given amount by the rate per unit expressed decimally.

EXAMPLE 1.—What is the commission on $790·80 at 3 per cent.?
Ans. $790·80 × ·03 = $23·724.

EXAMPLE 2.—A commission merchant sells goods to the amount of $7982·75; what is his commission at 2¾ per cent.?
Ans. $7982·75 × ·0275 = 219·525625.

EXERCISE 95.

1. What is the commission on $1000 at 4½ per cent.? *Ans.* $45.
2. What is the commission on $1678·30 at 2¼ per cent.? *Ans.* $37·76175.
3. What is the commission on $7531·19 at 3¾ per cent.? *Ans.* $282·419625.
4. Find the commission on $508·60 at 1¼ per cent. *Ans.* $6·3575.
5. Find the commission on $7862·50 at 1¾ per cent. *Ans.* $137·61125.
6. An agent collects debts to the amount of $878.30; what is his commission at 2½ per cent.? *Ans.* $21·9575.
7. A correspondent purchases teas for me to the amount of $7193·16; what have I to pay him for commission at 3⅛ per cent.? $224·78625.
8. A commission merchant sells goods to the amount of $6734·10; what is his commission at 17 per cent.? *Ans.* $1144·797.
9. An agent sells 718 barrels of flour at $7·13 a barrel; what is his commission at 4¼ per cent.? *Ans.* $217·57195.
10. A commission merchant disposes of 8243 bushels of wheat at $1·85 per bushel; what is the amount of his commission at 5⅝ per cent. *Ans.* $857.7871875.

BROKERAGE.

6. Brokerage is the *percentage* charged by money dealers, called *Brokers*, for negotiating *notes*, *mortgages*, *bills of exchange*, &c., or for buying or selling stocks, &c.

7. Brokerage is merely another name for commission, and is computed by the same rule.

Exercise 96.

1. What is the brokerage on $7893·87 at 2 per cent.? *Ans.* $157.8774.
2. What is the brokerage on $8000 at ⅞ per cent.? *Ans.* $70.
3. What is the brokerage on $8643·22 at 1¼ per cent.? *Ans.* $108·04025.
4. What is the brokerage on $78963·80 at ⅞ per cent.? *Ans.* $690·93325.
5. What is the brokerage on $1987·27 at 3¾ per cent.? *Ans.* $74·522625.

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8. Commission and Brokerage should both be computed on the amount of money *collected* or *invested.*

For example: If I receive $10000 to invest, and charge 5 per cent., my brokerage would be $500 if I invested the whole $10000; but if, as is usually the case, I am requested to deduct, from the amount sent, my brokerage or commission, and invest the remainder, it would obviously be unjust to charge commission on the whole amount—i. e., on the sum invested and also on the sum I retain for commission. Hence, in all cases, the sum actually expended is the proper basis upon which to compute the commission, brokerage, &c.

9. To compute commission or brokerage when it is to be deducted in advance from a given amount, and the balance invested:—

RULE.

1. *Divide the given amount by* $1, *plus the commission on* $1, *and the result will be the sum to be invested.*
2. *Subtract the part to be invested from the given amount, and the remainder will be the commission or brokerage.*

Example.—A correspondent receives $16782, with instructions to deduct his commission at 3½ per cent., and invest the balance in sugar at 9½ cents per pound. How much sugar does he ship to his employer, and what is his commission?

OPERATION.

$16782 ÷ 1·035 = $16214·49275 = sum to be invested.
$16782 − $16214·49275 = $567·50725 = commission.
$16214·49275 ÷ 9½ cents = 170678·871 lbs. *Ans.*

Explanation.—The commission on $1, at the rate of 3½ per cent., is $0·035. Hence, for every time he receives $1·035, he keeps $0·035 for commission, and invests $1. It is plain, then, that if we divide the given amount, $16782, by $1·035, or, in other words, find how often the latter sum is contained in the former, we shall find how often he invests $1; i. e., how many dollars he invests.

The work may be proved by finding the commission on the sum invested (Art. 5), and comparing it with the commission as found by deducting the sum invested from the whole sum sent. If these are equal, the work is correct.

EXERCISE 97.

1. An agent receives $4000, with instructions to purchase Great Western Railway Stock. After deducting his brokerage at $1\frac{1}{4}$ per cent., how much money had he to invest, and what was his brokerage? *Ans.* Invested $3950·61728.
Commission $49·38271.
2. A merchant sends his agent $7500, with instructions to deduct his commission at $4\frac{1}{2}$ per cent., and purchase laces with the remainder. What is the commission, and what sum was expended in laces? *Ans.* Commission $322·96651.
Invested $7177·03349.
3. A commission merchant receives $8470, with instructions to purchase the best brand of Canadian superfine flour at $6·40 per barrel. He is to receive out of this sum 5 per cent. on the amount he invests. How many barrels of flour does he purchase? *Ans.* $1260\frac{5}{12}$ barrels.
4. A broker receives $11000, with instructions to invest it in Bank stock—deducting his brokerage at $\frac{7}{8}$ per cent. What sum had he to invest? *Ans.* $10904·584882.
5. If I remit to my agent $13000, instructing him to purchase broad cloth at $3.63 per yard, and he keeps $4\frac{1}{2}$ per cent. on the sum invested, for commission; how much cloth does he send me, and what is his commission? *Ans.* 3427·0499 yards of cloth.
$559·8086 commission.

STOCK.

10. Stock is a term used to denote the *Capital* of moneyed institutions, as Banks, Railroad Companies, Gas Companies, Insurance Companies, Manufactories, &c.

11. Stock is usually divided into portions of $100 or £100 each, called *shares*, and the different individuals owning these are called *shareholders* or *stockholders*.

12. The *Association of Shareholders* is called a *Company* or *Corporation;* and the Act of Parliament specifying their corporate powers, rights, and privileges is called a *charter*.

13. The *nominal* or *par* value of a share is its original cost of valuation.

14. The market or real value of a share is the sum for which it can be sold.

15. The rise and fall in the value of Stock is reckoned at a certain per cent. on its *nominal* or *par* value.

16. When stocks sell for their original cost or valuation, they are said to be at *par;* when they sell for more than their original valuation, they are said to be *at a premium or advance,* or *above par;* when they do not bring their original cost or valuation, they are said to be *at a discount,* or *below par.*

Note.—*Par* is a Latin word, and means *equal or a state of equality.* Stock is *at par* when a hundred-dollar share sells for $100; it is *above par* when it brings more than $100, and *below par* when it will not bring as much as $100.

17. Persons who deal in stocks are called *stock-brokers* or *stock-jobbers.*

18. To find how much stock either *above* or *below par* a given sum will purchase:—

RULE.

Divide the given amount by the worth of $1 stock, and the result will be the stock required.

Example 1.—How much stock at 10 per cent. *below par* can be purchased for $25000? *Ans.* $25000 ÷ 0·90 = $$27777·77\frac{7}{9}$.

Explanation.—When stock is 10 per cent. *below par*, each share of $100 sells for only $90, i. e. $90 money will purchase $100 stock, therefore $0·90 money will purchase $1 stock, and the given sum will purchase $1 stock as often as it (the given sum) contains $0·90.

Example 2.—How much stock at 15 per cent. *premium* may be purchased for $7000? *Ans.* $7000 ÷ 1·15 = $6086·9565.

Explanation.—When stock is 15 per cent. *above par*, it requires $115 money to purchase $100 stock, or $1·15 money to purchase $1 stock. Hence if we divide the whole sum to be invested by the value of $1 stock, it is evident we must get the amount of stock produced.

Example 3.—I own $16400 stock of the Bank of Montreal, and sell out at 13 per cent. premium. What do I receive? *Ans.* $16400 × 1·13 = $18532.

Explanation.—Each $100 stock brings me $113 money, or $1 stock brings $1·13 money, therefore $16400 stock must bring $16400 × 1·13 money.

Exercise 98.

1. A person has $9000 which he wishes to invest in Grand Trunk Railway shares, then selling at 17 per cent. discount, what amount of stock can he purchase? *Ans.* $10843·373.
2. If I invest $8500 in Upper Canada Bank stock, which is selling 11 per cent. *above par*, what amount of stock do I receive? *Ans.* $7657·6576.

3. If I remit to my agent $17500, with instructions to deduct his brokerage at 1¼ per cent., and invest the remainder in Great Western Railroad stock, then selling at 7 per cent. premium, what amount of stock do I receive? *Ans.* $16153·22.
4. If I receive $20000, with instructions to deduct my commission at 1¾ per cent., and invest the balance in stock, which is then selling at 3 per cent. discount, what amount of stock do I remit to my employer? *Ans.* $20263·987.
5. Mr. A. owns 200 shares in the Canada Life Assurance Company. The par value is $100 a share, the stock at a premium of 5½ per cent.; if I purchase it through a broker who charges me ⅞ per cent. for the transaction, how much do my 200 shares cost me? *Ans.* $21284·625.

INSURANCE.

19. Insurance is a written agreement by which an individual or an incorporated company becomes bound, in consideration of a certain sum paid in advance, to exempt the owners of certain kinds of property, as houses, household furniture, merchandise, ships, &c., from loss by fire, shipwreck, or other calamity.

20. The *Written Instrument*, or contract between the parties, is called a *Policy of Insurance.*

21. The sum paid for the insurance is called the *Premium*, and is usually a certain per cent. on the sum for which the property is insured.

22. Houses, merchandise, furniture, &c., are usually insured against risk of fire for the year, or other specified time.

NOTE.—The rate of insurance on dwelling houses, stores, goods, household furniture, &c., varies from ¼ to 2 per cent. per annum, on the sum insured according to the character and position of the tenement; vessels are insured for the voyage or the year.

23. To compute the premium for insurance for 1 year, or a specified time, we use the same rule as for Commission or Brokerage.

EXAMPLE.—If I insure my house and furniture for $7389, at the rate of 1¼ per cent. per annum, what premium must I pay yearly?

Ans. $7389 × ·0125 = $92·3625.

EXPLANATION.—1¼ per cent., i. e. $1·25 per $100, is equal to $0·0125 per dollar. The premium therefore will be as many times $0·0125 as the sum insured contains $1; i. e. the premium will be 0·0125 × 7389.

Exercise 99.

1. What is the premium for insurance on $7500, at $1\frac{3}{4}$ per cent.? *Ans.* $131·25.

2. What is the premium for insurance on $8375, at $\frac{3}{4}$ per cent.? *Ans.* $62·8125.

3. What is the premium for insurance on $6000, at $1\frac{7}{8}$ per cent.? *Ans.* $112·50.

4. What is the premium for insurance on $5000 at $1·17 per cent. (i. e. per $100)? *Ans.* $58·50.

5. What is the premium for insurance on $6400, at $0·90 per cent.? *Ans.* $57·60.

6. What is the premium for insurance on $4500, at $0·35 per cent.? *Ans.* $15·75.

7. What premium must I pay for insuring a cargo of flour worth $36000, from Quebec to Liverpool, at $3 per cent.? *Ans.* $1080.

8. A firm, owning four steamers running on Lake Ontario, effect an insurance with a company in Toronto to the amount of $27000 on each, paying $4·82 per cent. (i. e. $4\frac{82}{100}$ per cent.) What is the total premium on the four steamers? *Ans.* $5205·60.

9. What is the annual premium on an insurance for $39000, at $2\frac{1}{5}$ per cent.? *Ans.* $858.

10. A farmer insures his barns and their contents to the amount of $17800. What premium does he pay at $\frac{1}{2}$ per cent? *Ans.* $89.

11. A vessel running between Hamilton and Oswego is insured for $12350, at the rate of $1\frac{3}{7}$ per cent. per month. To what does the premium of insurance amount for 7 months, beginning with the 10th of April and ending with the 10th of November? *Ans.* $1235.

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24. To find what sum must be insured on property so that, if destroyed, its value and the premium may both be recovered—

RULE.

Divide the value of the property by $1, minus the premium on $1 at the given rate per cent.

Example 1.—A ship-owner wishes to insure a vessel valued at $17450, so that if it be wrecked he may recover both the value of the vessel and the premium. In order to do so, for what sum must he insure, at $4·60 per cent.? *Ans.* $17450÷·954=$18291·40461.

EXPLANATION.—If I insure goods to the value of $100, at 4·6 per cent., and they are destroyed, I receive only $95·40 towards my loss, since I paid $4·60 for insurance; that is, for every $1 of my loss I receive $0·954. Since, then, the recovery of $0·954 requires $1 to be insured, the recovery of $17450 will require as many dollars to be insured as $0·954 is contained times in $17450.

PROOF.—$18291·40461 × ·046 = $841·40461 = the premium, and $18291·40461 − $841·40461 = $17450 = value of the vessel.

EXAMPLE 2.—What sum must be insured on a house valued at $6000, at 3 per cent. so that in case of fire the value of both premium and property may be secured? *Ans.* $6000 ÷ ·97 = $6185·567.

EXPLANATION.—For every dollar I lose (taking premium into account) I receive 97 cents; that is, in order to receive 97 cents, I must insure for $1, and in order to receive $6000, without any loss, I must insure for $6000 ÷ ·97 = $6185·567.

EXERCISE 100.

1. For what sum must I insure a cargo valued at $17000, so that in case the whole is lost I may recover both the value of the property and the premium of 3½ per cent.? *Ans.* $17616·58.
2. For what sum must I insure on $22750 in order to cover both the premium of 6 per cent. and the value of the property insured? *Ans.* $24202·127.
3. What sum must be insured at 2¼ per cent. on property worth $15000 so that the owner may be secured against all loss? *Ans.* $15345·2685.
4. A steamer worth $33000 is insured at 5¾ per cent. for such a sum, that in case of its becoming a total wreck, the owners may recover both the worth of the vessel, and the premium of insurance. For what sum is it insured? *Ans.* $35013·2625.

CUSTOM HOUSE BUSINESS.

25. All goods coming into Canada from Foreign countries are required by law to be landed at certain places or ports called *Ports of Entry.*

26. At every Port of Entry in Canada, the Government has an establishment called a *Custom House*, with one or more officers attached to it, called *Custom-House Officers.*

27. A certain charge called a *Duty*, fixed by Act of Parliament, is made upon nearly all goods entering Canada from Foreign countries.

28. It is the business of the Custom-House Officers to inspect the cargoes of all vessels entering at any of these ports, to examine the invoice of goods, collect the duties, &c., &c.

29. Besides the duties on merchandise, all vessels engaged in commerce are required to pay certain charges for the privilege of entering the port, &c.; these charges are called harbor dues.

30. The duties levied by law on goods imported into Canada are of two kinds:

1st. Specific duties.
2nd. Ad Valorem duties.

31. A specific duty is a certain sum levied on the ton, cwt., lb., gallon, square yard, &c., of a particular kind of merchandise, as so much per square yard on woollens, flannels or cloths, so much per lb. on tea, so much per gallon on brandy, wine, &c.

32. An ad valorem duty is a certain percentage on the actual cost of the goods in the country in which they were purchased.

Thus an ad valorem duty of 10 per cent. on satin purchased in France is a charge for duty at 10 per cent. of the sum the invoice of satin cost in France.

Note 1.—The term *ad valorem* is from the Latin; and means *according to the value*, i. e., *upon the value*.

Note 2.—An invoice is a written statement of the goods, showing the quantity of each sort and its value or price.

33. In the United States Custom Houses certain legal allowances are made for draft, tare, leakage, &c., before specific duties are imposed. In Canada, however, as before remarked, (Art. 4, Sect. VI.,) these are not known, the tare being found by actually weighing one or more of the boxes, &c., containing the goods, and the leakage by gauging the cask.

Note.—At present (1859) the various kinds of spirits are the only articles upon which specific duties are charged by the Canadian Tariff.

34. To calculate the specific duty on an invoice of goods—

RULE.

Deduct the tare, leakage, &c., and multiply the remainder by the given duty per gallon, lb., yard, &c.

Example 1. At 4¼ cents per lb. what is the specific duty on 7 bags of coffee weighing 73 lbs., each, allowing 4 lbs. per 100 or tare?

OPERATION.

$78 \times 7 = 511$ lbs. = gross weight.
$511 \times \cdot 04 = 20\frac{11}{25}$ lbs. = tare.

$490\frac{14}{25}$ = net at $4\frac{1}{4}$ cents per lb. $= 490\frac{14}{25} \times 4\frac{1}{4} = \$20 \cdot 8488.$ = *Ans.*

Example 2.—What is the specific duty on 10 chests of tea, the net weight 783 lbs., at 11 cents per lb.?

OPERATION.

$783 \times 11 = 8613$ cents $= \$86 \cdot 13$. *Ans.*

Exercise 101.

1. What is the specific duty, at $3\frac{1}{2}$ cents per lb., on 5 hhds. of sugar, each weighing 1347 lbs., allowing tare 6 lbs. per 100? *Ans.* $221·58.
2. What is the specific duty, at $1·20 per 100 lbs., on 11 bags of rice, each weighing 127 lbs., allowing 3 lbs. per 100 for tare? *Ans.* $16·26.
3. What is the specific duty, at 13 cents per gallon, on 129 gallons of oil? *Ans.* $16·77.
4. What is the specific duty, at $5\frac{3}{4}$ cents per lb., on 207 drums of figs, each weighing 31 lbs., allowing $2\frac{1}{4}$ lbs. a drum for tare? *Ans.* $342·1968.
5. What is the specific duty, at 47 cents per yard, on 214 yards of black silk velvet? *Ans.* $100·58.

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35. To find the *ad valorem* duty on an invoice of merchandise—

RULE.

Multiply the value of the goods at the place in which they were purchased by the per cent. charged, expressed decimally, and the result will be the duty required.

Example 1.—What is the ad valorem duty, at 27 per cent. on an invoice of brandy which cost $7493·70?

OPERATION.

$\$7493 \cdot 70 \times \cdot 27 = \$2023 \cdot 299$. *Ans.*

Example 2.—What is the ad valorem duty, at 19 per cent. on a quantity of broadcloth which cost $4116·40?

OPERATION.

$\$4116 \cdot 40 \times \cdot 19 = \$782 \cdot 116$. *Ans.*

Exercise 102.

1. What is the ad valorem duty, at 21 per cent. on an invoice of silks which cost $17429·80? *Ans.* $3660·2580.
2. What is the ad valorem duty, at $7\frac{1}{2}$ per cent. on 40 boxes of tea which cost $2920·16? *Ans.* $219·012.

3. What is the ad valorem duty, at 25 per cent. on an invoice of jewelry which cost $71342·90? *Ans.* $17835·725.
4. What is the ad valorem duty, at 20 per cent. on an invoice of boots and shoes which cost $913·73? *Ans.* $182·746.
5. What is the ad valorem duty at 33 per cent. on an invoice of French silks which cost $14713·19? *Ans.* $4855·3527.

ASSESSMENT OF TAXES.

36. A tax is a certain sum required to be raised by a municipality for local improvement, payment of officers, and other general purposes. It is collected from each citizen in proportion to the value of his property.

37. In levying taxes the first thing to be done is to make a complete inventory of the value of all the property in the city, town, township, &c., in which the tax is to be raised. This inventory is made by officers called Assessors appointed by the municipality.

38. To calculate the amount of taxes any one individual has to pay—

RULE.

Divide the whole sum to be levied by the whole value of rateable property in the town, township, &c.: the quotient will be the sum to be paid on each dollar.

Multiply the rate per dollar by the amount of the person's property, and the product will be the amount of his tax.

EXAMPLE.—A certain township requires to raise the sum of $14729·00 for general purposes; the whole amount of rateable property in the municipality being set down at $2743500, what proportion must I bear if my property is assessed at $7490·00?

OPERATION.

$14729 ÷ $2743500 = $0·005368 = rate per dollar.
$0·005368 × 7490 = $40·20632. *Ans.*

EXERCISE 108.

1. The assessment rolls of a town show the value of the rateable property to be $7142300. A tax of $23900 is to be levied for general purposes; how much is my proportion, my property being set down at $14729·50? *Ans.* $49·2878.
2. A tax of $100000 is to be levied on a county having rateable property to the value of $5793000; what is the amount borne by A, whose property is valued at $18600? *Ans.* $321·0732.

3. In the last example what would be the amount of B's tax, the value of his property being $7500? *Ans.* 129·465.
4. In the same example what would be the amount of C's tax, his property being assessed at $11400. *Ans.* $196·7868.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*The numerals after the Questions refer to the numbered articles of the Section.*

1. What is the meaning and derivation of the term per cent.? (1)
2. When the rate per cent. is known, how is the rate per unit obtained? (2)
3. How do we ascertain the percentage on any given number? (3)
4. What is commission? (4)
5. What is the person who sells goods for another called? (4)
6. How do we find the commission on any given sum? (5)
7. What is brokerage? (6)
8. How is the brokerage on any sum computed? (7)
9. Upon what sum should commission and brokerage be computed? (8)
10. Explain this by an example.
11. How do we compute commission or brokerage when it is to be deducted in advance from a given amount, and the balance invested? (9)
12. How is this rule proved? (9)
13. What is understood by the term Stock? (10)
14. How is Stock usually divided? (11)
15. What is meant by the terms Shareholders, Corporation, and Charter? (11 and 12)
16. What do you understand by the *nominal* or *par* value of Stock? (13)
17. What is meant by the *market* or *real* value of Stock? (14)
18. When is Stock said to be *at par?* when at a *premium* or *above par?* and when at a *discount* or *below par?* (15)
19. What is the meaning of the term *par?* (16, note)
20. What are persons who deal in Stocks called? (17)
21. When Stock is either above or below par, how do we find how much of it a given sum will purchase? (18)
22. What is Insurance? (19)
23. What is a Policy of Insurance? (20)
24. What is meant by the Premium of Insurance? (21)
25. For what length of time is property *usually* insured? (22)
26. How do we compute the premium of insurance on any amount of goods, property, &c.? (23)
27. How do we compute the amount for which we must insure in order to cover both the value of the property and the premium paid? (24)
28. How may the truth of this rule be proved? (24)
29. What are Ports of Entry? (25)
30. What is the duty of Custom-House Officers? (26)
31. What are duties? (27)
32. What are harbor dues? (29)
33. What different kinds of duties are levied on goods in Canada? (30)
34. What are *specific duties?* (31)
35. What is an *ad valorem duty?* (32)
36. What is the meaning of the term *ad valorem?* (33)
37. What is an invoice? (32)
38. What is the rule for computing specific duties? (34)
39. What is the rule for calculating *ad valorem* duties? (35)
40. What is a tax? (36)
41. How are taxes imposed? (9 and 36)

SECTION VIII.

INTEREST, DISCOUNT, EQUATION OF PAYMENTS, AND PARTNERSHIP.

1. Interest is the sum allowed for the use of money, and is usually reckoned at a certain rate per cent. per annum; that is, so many pounds for the use of £100 for one year, so many dollars for the use of $100 for one year, &c.

Note.—The term *per cent.* means *per hundred; per annum* means *per year.*

2. Interest differs from Commission, Brokerage, &c., in that the latter are computed at a certain per cent. without regard to time, while interest is calculated at a certain rate per cent. for one year, and consequently for longer and shorter periods in like proportion.

3. The Principal is the sum lent.

4. The Rate *per cent.* is the sum paid for the use of each hundred dollars, pounds, &c.

5. The Rate *per unit* is the sum paid for the use of each dollar, pound, &c.

6. The Interest is the whole sum received for the use of the principal.

7. The Amount is the sum obtained by adding together the principal and the interest.

Thus, if I lend $200 for a year, on the agreement that I am to receive interest at the rate of 7 per cent. (*per annum*, understood), at the end of the year I receive back the $200, and in addition $14 for interest. Here,

$200·00 is the principal.
7·00 is the rate per cent.
0·07 is the rate per unit.
14·00 is the interest.
214·00 is the amount=principal + interest.

8. Interest is either Simple or Compound.

9. Money is lent at Simple Interest when the Interest is not added to the principal so as so bear interest.

Thus, if $100 be lent at simple interest at 5 per cent., the *principal* remains unchanged, being always $100, and the *interest* for each successive year is $5.

10. Money is lent at Compound Interest when the interest, as it falls due from time to time, is added to the principal; the sum thus obtained constituting a *new* principal for the ensuing year, half year, quarter, &c., as the case may be.

Thus, if $100 be lent at 5 per cent. per annum compound interest, the principal changes at the end of each year, being $100 for the first year, $105 (i. e. former principal + its interest) for the second, $110·25 for the third, &c. The interest is consequently $5 for the first year, $5·25 for the second, $5·5125 for the third, &c.

SIMPLE INTEREST.

11. Questions in Interest are dependent on Proportion, and may all readily be solved by one or more statements in the Rule of Three; but in order to deduce special rules, we shall represent the different quantities by their initial letters, and thus obtain a series of algebraic formulæ, which, translated, become the common arithmetical rules for interest.

It is to be presumed that the pupil has made sufficient progress in Algebra before he arrives at this point, to readily understand what follows. The operations involved are of the simplest kind, and may without difficulty be comprehended, even by those wholly ignorant in Algebra. The only part, however, absolutely necessary for working any problem in interest, is the *interpretation* of the *formula*, i. e. the *arithmetical rule*, and this we have always appended. A glance at the formulæ and the corresponding rules will show how much less labor is necessary to remember the former than the latter; and indeed the pupil should be required to deduce from time to time any formulæ he may find it necessary to use.

Note.—When two or more letters are written together thus, prt. the meaning is that the values of these letters are to be *multiplied* together. Thus, Prt means that the value of P is to be multiplied by the value of r, and that by the value of t.

When letters are written in the form of a fraction, thus $\frac{A-P}{Pr}$ the meaning is the same as in common arithmetical fractions; i. e., that the part constituting the numerator is to be divided by the part constituting the denominator.

Thus, $\frac{A-P}{Pr}$ means that the value of P is to be subtracted from the value of A, and this difference is to be divided by the value of P multiplied by the value of r.

12. Let P = *Principal*, I = *Interest*, A = *Amount*, r = rate per unit, and t = time (i. e., number of years).

$$I = Prt \ (I.)$$

$$P = \frac{I}{rt} \ (II.)$$

$$r = \frac{I}{Pt} \ (III.)$$

$$t = \frac{I}{Pr} \ (IV.)$$

$$A = P(1+rt) \ (V.)$$

$$P = \frac{A}{1+rt} \ (VI.)$$

$$r = \frac{A-P}{Pt} \ (VII.)$$

$$t = \frac{A-P}{Pr} \ (VIII.)$$

$$t = \frac{n-1}{r} \ (IX.)$$

$$r = \frac{n-1}{t} \ (X.)$$

$$n = tr+1 \ (XI.)$$

Then because r = interest of $1 for 1 year, and t = number of years, rt = interest of $1 for the given time, and Prt = interest of given principal for given time and at given rate. Therefore $I = Prt$ and dividing each of these equals, 1st by rt, 2nd by Pt, and 3rd by Pr, we get formulas (II.) (III.) and (IV.) in the margin.

Again, because rt = interest of $1 at given rate and for given time, $1+rt$ = the *amount* of $1 at given rate and time, and P times $1+rt$, that is, $P(1+rt)$ = amount of given principal at the given rate and time. Therefore $A = P(1+rt)$, which is formula (V.) in the margin, and dividing each of these equals by $1+rt$, we get formula (VI.) in the margin. Taking (V.) and actually multiplying as indicated, the part within the brackets by P, we get $A = P + Prt$; and subtracting P from each of these, we get $A-P = Prt$. Dividing these equals, 1st by Pt and 2nd by Pr, we get formulas (VII.) and (VIII.) in the margin.

Lastly, if we are required to find in what time any sum of money will amount to any given number of times itself at a given rate per cent., or, in other words, in what time any principal will amount to n times that principal where n simply stands for the required *number of times*, we have in formula (VIII.) in the margin,

$t = \frac{A-P}{Pr} = \frac{nP-P}{Pr}$, because the amount is to be nP;

and dividing both numerator and denominator of this fraction by P, we get formula (IX.) in the margin, multiplying (IX.) by r we get $tr = n-1$; and dividing these equals by t, we get formula (X.); and, again, adding 1 to each of these same equals, we get formula (XI.)

APPLICATIONS.

13. When the principal, rate per cent., and time are given, *to find the interest*—

Rule $I = Prt$ (i.)

Interpretation.—*The interest is found by multiplying the principal by the rate per unit, and the resulting product by the time.*

Example.—What is the interest on $342·20 for 7 years at 8 per cent.?

OPERATION.

Here $P = \$342·20$, $r = ·08$, and $t = 7$.
Then $I = Prt = \$342·20 \times ·08 \times 7 = \$191·632$. *Ans.*

14. When the interest, rate per cent., and time are given *to find the principal*—

Rule. $P = \frac{I}{rt}$ (ii.)

Interpretation.—*The principal is found by dividing the interest by the product of the rate per unit and the time.*

EXAMPLE.—What principal will give \$207·50 interest in $6\frac{1}{2}$ years at $4\frac{3}{4}$ per cent.?

OPERATION.

Here $I = \$207{\cdot}50$, $t = 6{\cdot}5$, and $r = {\cdot}0475$.

Then $P = \frac{I}{rt} = \frac{\$207{\cdot}50}{6{\cdot}5 \times {\cdot}0475} = \frac{\$207{\cdot}50}{{\cdot}30875} = \$672{\cdot}064$. *Ans.*

15. When the interest, principal, and time are given, *to find the rate per cent.*—

RULE. $r = \frac{I}{Pt}$ (iii.)

INTERPRETATION.—*The rate per unit is found by dividing the interest by the product of the principal and time, and the rate per cent. is found from the rate per unit by multiplying the latter by* 100.

EXAMPLE.—At what rate per cent. will \$729·18 give \$109·11 interest in 9 years?

OPERATION.

Here $P = \$729{\cdot}18$, $I = \$109{\cdot}11$, and $t = 9$.

Then $r = \frac{I}{Pt} = \frac{109{\cdot}11}{729{\cdot}18 \times 9} = \frac{109{\cdot}11}{6562{\cdot}62} = 0{\cdot}01662$ = rate per unit.

Therefore the rate per cent. $= 0{\cdot}01662 \times 100 = 1{\cdot}662 = 1\frac{2}{3}$ nearly. *Ans.*

16. When the interest, principal, and rate per cent. are given, *to find the time*—

RULE. $t = \frac{I}{Pr}$ (iv.)

INTERPRETATION.—*The time is found by dividing the interest by the product of the principal and rate per unit.*

EXAMPLE.—In what time will \$850 give \$89·75 interest, at 13 per cent.?

Here $P = \$850$, $I = \$89{\cdot}75$, and $r = {\cdot}18$.

Then $t = \frac{I}{Pr} = \frac{89.75}{850 \times {\cdot}13} = \frac{89{\cdot}75}{110{\cdot}5} = \frac{897{\cdot}5}{1105} = 0{\cdot}812217$ years = 9 months 22 days.

17. When the principal, rate per cent., and time are given, *to find the amount*—

RULE. $A = P(1 + rt)$ (v.)

INTERPRETATION.—*The amount is found by multiplying the principal by the amount of \$1 for the given rate and time.*

EXAMPLE.—To what sum will \$789·80 amount in 11 years, at 3 per cent.?

OPERATION.

Here $P = \$789{\cdot}80$, $r = {\cdot}03$, and $t = 11$.

Then $A = P(1 + rt) = \$739{\cdot}80 \times 1{\cdot}33 = 1050{\cdot}434$. *Ans.*

NOTE.—$(1 + rt)$ in this question $= 1 + {\cdot}03 \times 11 = 1 + {\cdot}33 = 1{\cdot}33$.

18. When the amount, rate per cent., and time are given, *to find the principal*—

Rule. $P = \dfrac{A}{1+rt}$ (vi.)

Interpretation.—*The principal is found by dividing the given amount by the amount of $1 for the given time at the given rate.*

Example.—What principal put to interest at 7½ per cent. will amount to $2000 in 8 years?

OPERATION.

Here $A = \$2000$, $r = {\cdot}075$ and $t = 8$.

$$\text{Then } P = \frac{A}{1+rt} = \frac{2000}{1{\cdot}60} = \frac{20000}{16} = \$1250.\ \textit{Ans.}$$

19. When the amount, principal, and time are given, *to find the rate per cent.*—

Rule. $r = \dfrac{A-P}{Pt}$ (vii.)

Interpretation.—*The rate per unit is found by subtracting the principal from the amount, and dividing the difference by the principal multiplied by the time. The rate per cent. is found by multiplying the rate per unit by* 100.

Example.—At what rate per cent. will $730 amount to $2783·80 in 23 years?

OPERATION.

Here $A = \$2783{\cdot}80$, $P = \$730$ and $t = 23$.

$$\text{Then } r = \frac{A-P}{Pt} = \frac{\$2783{\cdot}80-\$730}{\$730 \times 23} = \frac{\$2053{\cdot}80}{\$16790} = {\cdot}1223 = \text{rate per unit.}$$

Hence rate per cent. = 12·23 = 12¼ nearly.

20. When the amount, principal, and rate per cent. are given, *to find the time*—

Rule. $t = \dfrac{A-P}{Pr}$ (viii.)

Interpretation.—*The time is found by subtracting the principal from the amount, and dividing the difference by the principal multiplied by the rate per unit.*

Example.—In what time will $666·33 amount to $983·73 at 12 per cent.?

OPERATION.

Here $A = \$983{\cdot}73$, $P = \$666{\cdot}33$ and $r = {\cdot}12$.

$$\text{Then } t = \frac{A-P}{Pr} = \frac{983{\cdot}73-666{\cdot}33}{666{\cdot}33 \times {\cdot}12} = \frac{317{\cdot}40}{79{\cdot}9596} = \frac{3174000}{799596} = 3{\cdot}9695 \text{ years} =$$

3 years 11 months 19 days. *Ans.*

21. *To find the time* in which any sum will amount to any given number of times itself at a given rate per cent.—

RULE. $t = \frac{n-1}{r}$ (ix.)

INTERPRETATION.—*To find the time in which a given sum will amount to* n *times itself at a given rate per cent., subtract* 1 *from* n, *and divide the remainder by the rate per unit.*

EXAMPLE 1.—In what time will any sum of money amount to *eleven* times itself at 8 per cent.?

OPERATION.

Here $n = 11$ and $r = \cdot 08$.

Then $t = \frac{n-1}{r} = \frac{11-1}{\cdot 08} = \frac{10}{\cdot 08} = \frac{1000}{8} = 125$ years. *Ans.*

EXAMPLE 2.—In what time will $67·83 *quadruple* itself at 4¾ per cent.?

OPERATION.

Here $n = 4$, since the money is to *quadruple* itself, and $r = \cdot 0475$.

Then $t = \frac{n-1}{r} = \frac{4-1}{\cdot 0475} = \frac{3}{\cdot 0475} = \frac{30000}{475} = 63\cdot 157$ years. *Ans.*

22. *To find the rate per cent.* at which any sum will amount to a given number of times itself in a given time—

RULE. $r = \frac{n-1}{t}$ (x.)

INTERPRETATION.—*The rate per unit is found by subtracting* 1 *from* n, *the number of times itself to which the given principal is to amount, and dividing the remainder by the given number of years.*

EXAMPLE.—At what rate per cent. will a given sum amount to 25 times itself in 72 years?

OPERATION.

Here $n = 25$, $t = 72$.

Then $r = \frac{n-1}{t} = \frac{25-1}{72} = \frac{24}{72} = \frac{1}{3} = \cdot 33\frac{1}{3}$ = rate per unit.

Hence rate per cent. $= 33\frac{1}{3}$. *Ans.*

23. *To find to how many times itself* a given sum will amount in a given time at a given rate per cent.—

RULE. $n = tr + 1$ (xi.)

INTERPRETATION.—*The number of times, or* n, *is found by multiplying the time by the rate per unit, and adding* 1 *to the product.*

EXAMPLE.—To how many times itself will *four cents* amount in 20 years at 17 per cent.?

OPERATION.

Here $t = 20$ and $r = \cdot17$.
Then $n = tr + 1 = 20 \times \cdot17 + 1 = 3\cdot4 + 1 = 4\cdot4 = 4\frac{2}{5}$ times itself. *Ans.*

EXERCISE 104.

1. What is the interest on $723·19 for 7·32 years, at 6·7 per cent.? *Ans.* $354·6813036.
2. To what sum will $857·19 amount in 6½ years, at 6½ per cent.? *Ans.* $1219·352775.
3. To how many times itself will £2 19s. 9½d. amount in 11 years, at 72½ per cent.? *Ans.* 8·975, or nearly 9 times.
4. In what time will $654·32 give $234·56 interest, at 7 per cent.? *Ans.* 5·12112, or 5 years 1 m. 13 days.
5. At what rate per cent. will $700 amount to $1200 in 5 years? *Ans.* 14$\frac{2}{7}$ per cent.
6. In what time will any sum of money quadruple itself, at 23 per cent.? *Ans.* 13 years 15 days.
7. Find the time in which $270 will give $87 interest, at 7 per cent. *Ans.* 4 years 7$\frac{5}{21}$ months.
8. To what sum will $680 amount in 11½ years, at 11 per cent.? *Ans.* $1540·20.
9. What principal will amount to $2000 in 20 years, at 8 per cent.? *Ans.* $769·23$\frac{1}{13}$.
10. At what rate per cent. will any sum of money amount to 21 times itself in 24 years? *Ans.* $83⅓ per cent.
11. In what time will a given sum of money amount to 23 times itself, at 16 per cent.? *Ans.* 137½ years.
12. Find the interest on $679·18 at 7¾ per cent., for 11·73 years. *Ans.* $617·4255.
13. At what rate per cent. will $950 amount to $1763·42 in 10 years? *Ans.* 8·562 per cent., or rather over 8½ per cent.
14. In what time will $666 amount to $1347·50, at 6 per cent.? *Ans.* 17·054 + years, or 17 years 19 days.
15. In what time will $273 give $100 interest, at 9 per cent.? *Ans.* 4 years 25 days.
16. At what rate per cent. will $476·30 amount to $500 in 2 years? *Ans.* 2[illegible] per cent.
17. At what rate per cent. will $749·49 give $257 interest in 7 years? *Ans* 4·898 per cent.
18. What principal will amount to $1111·11 in 11 years, at 11 per cent? *Ans.* $502·7647.
19. Find the interest on £167·47, at 11 per cent. for 9 years. *Ans.* £165 15s. 10[illegible]d.

SPECIAL RULES.

24. The interest of $100 at 6 per cent., for one year, is $6; hence the interest on $1 at 6 per cent., for one year, is $0·06, and *for two months* it is ⅙ of $0·06; i. e., 1 *cent*.

Hence, to find the interest of $1, at 6 per cent. per annum for any number of months, we deduce the following—

RULE.

Divide the number of months by 2, and call the quotient cents.

Example 1.—What is the interest of $1 at 6 per cent. for 7 years and 9 months?

OPERATION.

7 years and 9 months=93 months, and 93÷2=46½ cents=$0·465. *Ans.*

Example 2.—Find the interest on $72·93 for 7 years and 8 months at 6 per cent.

OPERATION.

7 years 8 mo.=92 months, half of 92=46 cents=interest of $1 for given rate and time.

Then $0·46 × 72·93=$33·5478. *Ans.*

Exercise 105.

1. Find the interest of $1 for 11 months at 6 per cent. *Ans.* 5½ cents.
2. Find the interest on $1 for 16 months at 6 per cent. *Ans.* $0·08, or 8 cents.
3. Find the interest on $1 for 9 years 8 months at 6 per cent. *Ans.* $0·58.
4. What is the interest on $1 for 16 years 3 months at 6 per cent.? *Ans.* $·97½.
5. What is the interest on $1 for 11 years 7 months at 6 per cent.? *Ans.* $0·695.
6. What is the interest on $1 for 12 years 5 months at 6 per cent? *Ans.* $0·745.
7. Find the interest on $279·40 for 3 years 2 months at 6 per cent. *Ans.* $53·086.
8. Find the interest on $189·70 for 6 years 7 months at 6 per cent. *Ans.* $74·9315.
9. Find the interest on $1463 for 3 years 11 months at 6 per cent. *Ans.* $343·805.
10. Find the interest on $28967·50 for 11 years 1 month at 6 per cent. *Ans.* $19263·3875.

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25. Since in computing interest the month is taken as 30 days, two months will contain 60 days, and, by Art. 24, the interest on $1 at 6 per cent. for 2 months or 60 *days* is one cent, the interest on $1 at 6 per cent. per annum, for 6 *days*, will therefore be $\frac{1}{10}$ of one cent; i. e. *one mill* or $\frac{1}{1000}$ of $1.

Hence, to find the interest on $1 at 6 per cent. per annum for days, we have the following—

RULE.*

Call one-sixth of the number of days mills or thousandths of a dollar.

EXAMPLE.—What is the interest on $1 at 6 per cent. for 16 days?

OPERATION.

16÷6=2⅔ mills=$0·0026. *Ans.*

EXERCISE 106.

1. What is the interest on $1 for 2 days at 6 per cent.? *Ans.* $0·0003.
2. What is the interest on $1 for 7 days at 6 per cent.? *Ans.* $0·001⅙.
3. What is the interest on $1 for 11 days at 6 per cent.? *Ans.* $0·001⅚.
4. What is the interest on $1 for 27 days at 6 per cent.? *Ans.* $0·004½.
5. What is the interest on $1 for 47 days at 6 per cent.? *Ans.* $0·007⅚.
6. Required the interest on $1 for 8 months 12 days at 6 per cent. *Ans.* $0·042.
7. Required the interest on $1 for 66 days at 6 per cent. *Ans.* $0·011.
8. Required the interest on $1 for 2 years 2 months 19 days at 6 per cent. *Ans.* $0·133⅙.
9. Find the interest on $1 for 7 years 8 months 9 days at 6 per cent. *Ans.* $0·461½.
10. What is the interest on $1 for 17 years 11 months 23 days at 6 per cent.? *Ans.* $1·078⅚.
11. Required the interest on $1 for 12 years 7 months 17 days at 6 per cent. *Ans.* 0·757⅚.

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26. To find the interest on any sum of money at 6 per cent. per annum for any time—

RULE.

Find the interest on $1 for the given time, by Arts. 24 *and* 25, *and multiply this by the given principal.*

EXAMPLE.—What is the interest on $763·20 at 6 per cent. for 6 years 7 months and 26 days?

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[*] This is the method in common use for computing interest for days: but, since it considers the year as containing only 360 days instead of 365, the result is too large by $\frac{5}{365}$, or $\frac{1}{73}$ of itself. Hence, when perfect accuracy is desired, the interest for the days when obtained by the rule must be diminished by $\frac{1}{73}$ part of itself.

OPERATION.

Interest on $1 for 6 years 7 months = $0·895
Interest on $1 for 26 days = 4⅓

Therefore interest on $1 for 6 yrs. 7 months 26 days = $0·899⅓
Then* $0·899⅓ × 768·20 = $304·7712. *Ans.*

EXERCISE 107.

1. Find the interest on $917·30 for 7 months 17 days at 6 per cent. *Ans.* $34·704516.
2. Find the interest on $842·50 for 3 months 13 days at 6 per cent. *Ans.* $14·462916.
3. Required the interest on $573·83 at 6 per cent. for 2 years 11 months 10 days. *Ans.* $101·3766.
4. Required the interest on $642·30 at 6 per cent. for 6 years 9 months 19 days. *Ans.* $262·16545.
5. Required the interest on $1427·87½ at 6 per cent. for 5 years 5 months 7 days. *Ans.* 465·7252.
6. Find the interest on $709·63 for 4 years 7 months 16 days at 6 per cent. *Ans.* $197·040596.
7. Find the amount of $2463·20 at 6 per cent. for 7 years 7 months 22 days. *Ans.* $3592·9877.
8. What is the interest on $999·99 at 6 per cent. for 9 years 9 months 9 days? *Ans.* $586·494135.
9. What is the interest on $68·70 for 3 years 4 months 27 days at 6 per cent.? *Ans.* $14·04915.
10. Find the interest on $742·63 at 6 per cent. for 3 years 28 days. *Ans.* $137·139.
11. To what sum will $200 amount in 7 years 4 months 11 days at 6 per cent.? *Ans.* 288·366.
12. To what sum will $743·63 amount in 9 years 3 months 9 days at 6 per cent.? *Ans.* $1157·460095.

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27. To find the interest on any sum at any other rate per cent. for any given time—

RULE.

Find the interest on the given principal for the given time at 6 per cent. by Art. 26.

Then add to or subtract from this interest such a fractional part of itself as the given rate exceeds or falls short of 6 *per cent. per annum.*

The amount is obtained by adding the interest and the principal together.

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* In order to obtain the correct answer, this fraction when it occurs must be retained in the form of a *vulgar fraction;* and in that case it is better to make the interest of $1 for the given time the *multiplier.*

EXAMPLE.—What is the interest on $450 for 3 years 6 months 11 days at 3 per cent.?

OPERATION.

Interest on $1 at 6 per cent. for given time=$0·211⅚.
Interest on $450 at 6 per cent. for given time=$0·211⅚ × 450=$95·325.
Hence interest on $450 at 8 per cent. for given time=$95·325+*one third* of $95·325=$127·10. *Ans.*

NOTE.—Since 8=6+2=6+⅓ of 6 we find the interest at 6 per cent., and increase it by *one third* of itself for the interest at 8 per cent.

So for interest at 9 per cent., we should find the interest at 6 per cent., and increase it by *one-half* of itself; for 7 per cent., increase the interest at 6 per cent. by *one-sixth;* at 14 per cent., double the interest at 6 per cent., and increase it by ⅓ of the interest at 6 per cent.; at 5 per cent., find the interest at 6 per cent. and deduct *one-sixth;* at 4½ per cent., find the interest at 6 per cent., and deduct *one-fourth*, &c., &c.

EXERCISE 108.

1. Required the interest on $1284·56 for 8 years 9 months 10 days at 7 per cent. *Ans.* $758·5685.
2. Required the interest on $9876·54 for 2 years 1 month 11 days at 3 per cent. *Ans.* $626·337245.
3. Required the interest on $715·30 for 3 years 7 months 10 days at 8 per cent. *Ans.* $206·6422.
4. To what sum will $555·55 amount in 2 years 4 months 8 days at 12 per cent.? *Ans.* $712·58546.
5. To what sum will $7766·55 amount in 100 days at 5 per cent.? *Ans.* $7874·41875.
6. To what sum will $500 amount in 8 years 8 months 8 days at 16 per cent.? *Ans.* $1195·111.
7. What is the interest on $576 for 3 years 5 months 7 days at 5 per cent.? *Ans.* $98·96.
8. What is the interest on $2478·91 for 2 years 6 months 11 days at 4½ per cent.? *Ans.* $282·285.
9. What is the interest on $780 from May 9, to December 11, at 6 per cent.? *Ans.* $28·08.
10. What is the interest on a note of $1830·63 from August 16, 1851, to June 19, 1852, at 7 per cent.? *Ans.* $109·63439.
11. What is the amount of a note of $6200 from Sept. 3, 1858, to January 9, 1859, at 6 per cent.? *Ans.* $6332·266.

PARTIAL PAYMENTS.

28. To compute the interest, on notes or bonds, when partial payments have been made—

RULE.

If the interest be paid by days:

Multiply the sum by the number of days which have elapsed before any payment was made. Subtract the first payment, and multiply *the remainder by the number of days which passed between the first and second payments. Subtract the second payment, and multiply this remainder by the number of days which passed between the second and third payments. Subtract the third payment, &c.*

Add all the products together, and find the interest of their sum for one day.

If the interest is to be paid by the week or month, substitute weeks or months for days, in the above rule.

EXAMPLE.—How much principal and interest have I to pay on the following note on the 10th November, 1859?

TORONTO, 18*th October*, 1858.

For value received, I promise to pay Timothy Thomas, or order, the sum of six hundred and twenty dollars, on demand, with interest at 6 per cent.

THOMAS WILLIAMS.

The following endorsements were made on this note:—

1858.—November 25th, there was endorsed $ 47·50
" December 28th, " " " 108·93
1859.—February 11th, " " " 216·18
" June 6th, " " " 60·10
" September 2nd, " " " 183·25

OPERATION.

From 18th October to 25th November there are 38 days.
" 25th Nov. to 28th December " 33 "
" 28th Dec. to 11th February " 45 "
" 11th February to 6th June " 115 "
" 6th June to 2nd September " 88 "
" 2nd September to 10th Nov. " 69 "

Whole sum $620·00 for 38 days = $23560·00 for 1 day.
First endorsement 47·50

Balance $572·50 for 33 days = $18892·50 for 1 day.
Second endorsement 108·93

Balance $463·57 for 45 days = $20860·65 for 1 day.
Third endorsement 216·18

Balance $247·39 for 115 days = $28449·85 for 1 day.
Fourth endorsement 60·10

Balance $187·29 for 88 days = $16481·52 for 1 day.
Fifth endorsement 183·25

Balance $4·04 for 69 days = 278·76 for 1 day.

Whole interest = that of $108523·28 for 1 day.
Interest on $108523·28 at 6 per cent. for 1 year = $6511·3968
Hence interest for 1 day = $6511·3968 ÷ 365 = $17·8394
Then interest due .... = $17·8394
Balance on note .... = 4·04

Principal and interest due = $21·8794

Exercise 109.

1. What principal and interest was due on the following note on the 7th October, 1860?

Guelph, *June 2nd*, 1859.

For value received, I promise to pay, on demand, to James George, or order, the sum of twelve hundred and seventeen dollars and thirty cents, with interest from date at 6 per cent.

Joseph Johns.

On this note there were endorsed the following payments:—

1859.—July 17th,	received	$207·80
" Oct. 6th,	"	209·60
" Dec. 11th,	"	320·90
1860.—March 29th,	"	421·83

Ans. $98·6816.

2. What principal and interest was due on the following note on the 1st May, 1863?

Port Hope, *June 17th*, 1860.

For value received, I promise to pay, on demand, to Messrs. Henly & Jobson, or order, the sum of seven thousand, three hundred and forty-eight dollars and twenty-five cents, with interest from date at 8 per cent.

Henry Goodpay.

On this note there were endorsed the following payments:

1860.—September 5th,	received	$2468·80
" December 7th,	"	392·20
1861.—June 11th,	"	982·20
1862.—February 7th,	"	2842·90
" December 19th,	"	317·23

Ans. $1003·1333.

COMPOUND INTEREST.

29. In the present article we shall merely take some of the simpler problems in Compound Interest, leaving the full discussion of the rule until after the pupil is familiar with the use of Logarithms. (See Sect. XI.)

30. We have seen (Art. 10) that when money is lent at compound interest, the interest is added to the principal at the close of each period, and, with it, constitutes a new principal for the next term.

Hence to find the compound interest of any sum for any given time at a given rate per cent.:—

RULE.

Find the interest on the given principal for one period, *i. e.*, ONE YEAR, HALF YEAR, *or* QUARTER, *as the case may be, and add it to the principal.*

Then find the interest on this amount for the NEXT PERIOD *and add it to the principal used for that period, as before.*

Proceed in this manner with each successive year or period of the proposed time.

Then the last result will be the amount of the given principal, at the given rate, for the given time. Subtract the given principal from this, and the remainder will be the Compound Interest required.

EXAMPLE.—What is the Compound Interest on $1000 for 4 years at 5 per cent. per annum?

OPERATION.

$1000·00	Principal.
50·00	Interest for 1st year.
$1050·00	Amount for 1 year = principal for 2nd year.
52·50	Interest for 2nd year.
$1102·50	Amount for 2 years = principal for 3rd year.
55·125	Interest for 3rd year.
$1157·625	Amount for 3 years = principal for 4th year.
57·88125	Interest for 4th year.
$1215·50625	Amount for 4 years.
1000·00	Given Principal.
Ans. $215·50625	= Compound Interest required.

EXERCISE 110.

1. What is the Compound Interest of $1800 for 5 years at 6 per cent. per annum? *Ans.* $608·806.
2. What is the Compound Interest of $700 for 3½ years at 7 per cent. half-yearly? *Ans.* $424·040.

NOTE.—Since the payments are made *half-yearly*, and bear interest at the rate of 7 per cent. per half year, we simply find the amount of the given principal at 7 per cent. for 7 payments.

3. What are the amount and Compound Interest of $673·40 for 2 years at 3 per cent. quarterly?
Ans. $853·0429 = Amount. $179·6429 = Interest.
4. What are the amount and Compound Interest of $860 for 3 years at 4 per cent. half-yearly?
Ans. $1088·1743 = Amount. $228·1743 = Interest.

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31. Compound Interest is most expeditiously calculated by the following—

TABLE

SHOWING THE AMOUNTS OF $1 OR £1 AT COMPOUND INTEREST, FOR ANY NUMBER OF PAYMENTS FROM 1 TO 50.

No. of Payments.	3 per cent.	4 per cent.	5 per cent.	6 per cent.	No. of Payments.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	1·03000	1·04000	1·05000	1·06000	26	2·15659	2·77247	3·55567	4·54938
2	1·06090	1·08160	1·10250	1·12360	27	2·22129	2·88337	3·73346	4·82235
3	1·09273	1·12486	1·15762	1·19102	28	2·28793	2·99870	3·92013	5·11169
4	1·12551	1·16986	1·21551	1·26248	29	2·35657	3·11865	4·11614	5·41839
5	1·15927	1·21665	1·27628	1·33823	30	2·42726	3·24340	4·32194	5·74349
6	1·19405	1·26532	1·34010	1·41852	31	2·50008	3·37313	4·53804	6·08810
7	1·22987	1·31593	1·40710	1·50363	32	2·57508	3·50806	4·76494	6·45339
8	1·26677	1·36857	1·47745	1·59385	33	2·65233	3·64838	5·00319	6·84059
9	1·30477	1·42331	1·55133	1·68948	34	2·73190	3·79432	5·25335	7·25102
10	1·34392	1·48024	1·62889	1·79085	35	2·81386	3·94609	5·51601	7·68609
11	1·38423	1·53945	1·71034	1·89830	36	2·89828	4·10393	5·79182	8·14725
12	1·42576	1·60103	1·79586	2·01220	37	2·98523	4·26809	6·08141	8·63609
13	1·46853	1·66507	1·88565	2·13293	38	3·07478	4·43881	6·38548	9·15425
14	1·51259	1·73168	1·97993	2·26090	39	3·16703	4·61637	6·70475	9·70351
15	1·55797	1·80094	2·07893	2·39656	40	3·26204	4·80102	7·03999	10·28572
16	1·60471	1·87298	2·18287	2·54035	41	3·35990	4·99306	7·39169	10·90286
17	1·65285	1·94790	2·29202	2·69277	42	3·46070	5·19278	7·76159	11·55703
18	1·70243	2·02582	2·40662	2·85434	43	3·56452	5·40049	8·14967	12·25045
19	1·75351	2·10685	2·52695	3·02560	44	3·67145	5·61651	8·55715	12·98548
20	1·80611	2·19112	2·65330	3·20713	45	3·78160	5·84118	8·98501	13·76461
21	1·86029	2·27877	2·78596	3·39956	46	3·89504	6·07482	9·43426	14·59049
22	1·91610	2·36992	2·92526	3·60354	47	4·01190	6·31782	9·90597	15·46592
23	1·97359	2·46472	3·07152	3·81975	48	4·13225	6·57053	10·40127	16·39387
24	2·03279	2·56330	3·22510	4·04893	49	4·25622	6·83335	10·92133	17·37700
25	2·09378	2·66584	3·38635	4·29187	50	4·38391	7·10668	11·46740	18·42515

32. To compute Compound Interest by the above Table:—

RULE.

Find by the table the amount of $1 for the given time and at the given rate.

Multiply the sum thus found by the given principal, and the result will be the required amount.

Subtract the principal from this amount, and the remainder will be the Compound Interest.

EXAMPLE 1.—What are the amount and Compound Interest of $3400 at 5 per cent. for 15 years?

OPERATION.

By the table the amount of $1 at 5 per cent. for 15 years = $2·07893.
Then $2·07893 × 3400 = $7068·362 = Amount.
3400 = Principal.

$3668·362 = Interest.

Example 2.—What is the amount and compound interest of £47 10s. for 6 years at 3 per cent. half yearly?

OPERATION.

£47 10s. = £47·5.

We find by the table that
£1·42576 is the amount of £1 for the given time and rate.
47·5 is the multiplier.

	£	s.	d.	
£67·7286 =	67	14	5¼	is the required amount.
	47	10	0	is the given principal.
And	£20	4	5¼	is the required interest.

Exercise 111.

1. What are the amount and compound interest on $875 for 11 years at 6 per cent.? *Ans.* Amount = $1661·0125.
Interest = $786·0125.

2. What are the amount and compound interest on $643·98 for 13 years at 4 per cent. half-yearly? *Ans.* Amount = $1785·41523.
Interest = $1141·43523.

3. What are the amount and compound interest of 1 cent at 6 per cent. per annum for 45 years? *Ans.* Amount = $·137646.
Interest = $·127646.

4. What are the amount and compound interest of $78·20 for 7 years at 3 per cent. quarterly? *Ans.* Amount = $178·916.
Interest = $100·716.

5. What are the amount and compound interest of $777·77 for 9 years at 5 per cent. half-yearly? *Ans.* Amount = $1871·7968.
Interest = $1094·0268.

6. What are the amount and compound interest of £44 5s. 9d. for 11 years at 6 per cent. per annum?
Ans. Amount = £84 1s. 5d.
Interest = £39 15s. 8d.

7. What are the amount and compound interest of £32 4s. 9¾d. for 3 years at 4 per cent. half-yearly?
Ans. Amount = £40 15s. 10¾d. nearly.
Interest = £8 11s. 1d.

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33. Given the amount, time and rate—to find the principal; that is, to find the *present worth* of any sum to be due hereafter—a certain rate of interest being allowed for the money now paid—

RULE.

Find by the Table the amount of $1 at a given rate and for the given time, and divide it into the given amount. The quotient will be the principal.

EXAMPLE.—What principal will amount to $10000 in 12 years at 6 per cent. compound interest?

OPERATION.

Amount of $1 for 12 years at six per cent. = $2·0122.
$10000 ÷ 2·0122 = $4969·684. *Ans.*

EXERCISE 112.

1. What principal will amount to $7439·87 in 7 years at 4 per cent. compound interest? *Ans.* $5653·697.
2. What principal will amount to $9193·90 in 20 years at 5 per cent. compound interest? *Ans.* $3465·081.
3. What ready money ought to be paid for a debt of £595 10s. $2\frac{3}{8}$d. to be due 3 years hence, allowing 6 per cent. per annum compound interest? *Ans.* £500.
4. What ready money ought to be paid for a debt of $7111·11, to be due 7 years hence, allowing 6 per cent. compound interest? *Ans.* $4729·295.
5. What principal, put to interest for 6 years, would amount to £268 0s. $4\frac{4}{5}$d. at 5 per cent. per annum? *Ans.* £200.

DISCOUNT.

34. Discount is an allowance made for payment of a debt before it is due.

35. The present worth of a debt payable at some future time, without interest, is that sum of money which, being put out at legal interest, *will amount* to the debt by the time it becomes due.

Thus, if I owe a man $100 and give him a note for that amount, payable one year hence without interest, the *present* value of my note is less than $100, since $100 being put out at interest for 1 year at 6 per cent. will amount to $106.

36. From Art. 18 it is evident that to find the present worth of a note, payable at some future time, without interest, is simply to find what principal, put to interest at the rate specified, will amount to the sum named on the face of the note in the given time; *i. e.* by the time the note becomes due.

Hence to find the present worth of any sum to be paid at some future time without interest, we have (Art. 18) the following:—

RULE. $P = \dfrac{A}{1 + rt}$

INTERPRETATION.—*The present worth is found by dividing the amount of the note, debt, &c., by the amount of $1, at the specified rate per cent. for the given time.*

NOTE.—*The discount is found by deducting the present value from the note, debt, &c.*

EXAMPLE 1.—What is the present value of a note for \$860 payable 3 years hence, allowing discount at the rate of 6 per cent. per annum?

OPERATION.

Here $A = \$860$, $r = \cdot06$, and $t = 3$. Whence $1 + rt = 1\cdot18$.

Then $P = \frac{A}{1 + rt} = \frac{860}{1\cdot18} = \$728\cdot81\frac{21}{59}$. *Ans.*

PROOF.—Interest on $728·81\frac{21}{59}$ for 3 years at 6 per cent. = $131·18\frac{38}{59}$.
Added principal.............................. = $728·81\frac{21}{59}$.
Amount.. = \$860·00

EXAMPLE 2.—What is the discount on a note for \$728·63 due 9 months hence, allowing discount at 7 per cent. per annum?

OPERATION.

Here $A = \$728\cdot63$, $r = \cdot07$, and $t = \cdot75$ year. Whence $1 + rt = 1\cdot0525$

Then $P = \frac{A}{1 + rt} = \frac{728\cdot63}{1\cdot0525} = \$692\cdot285$ present worth.

Then amount on face of note.. \$728·63
Present value........... 692·285
Discount........ \$ 36·344 *Ans.*

EXERCISE 113.

1. What is the present worth of a note for \$962, payable in one year, at 4 per cent. discount? *Ans.* \$925.
2. What is the present worth of \$2202, payable in 5 years and 9 months, at 6 per cent. per annum discount? *Ans.* \$1637·174.
3. What sum will discharge a debt of \$1003·50, to be due in 8 months hence, allowing 6 per cent. per annum discount? *Ans.* \$964·9038.
4. What ready money will now pay a debt of \$716 due 7 months hence, allowing discount at 8 per cent.? *Ans.* \$684·0764.
5. What ready money will now pay a debt of \$1342·50, due 125 days hence, at 6¼ per cent.? *Ans.* \$1313·266.
6. If a legacy of \$2400 is left to me on the 3rd of May, to be paid on the Christmas day following, what must I receive as present payment, allowing 5 per cent. per annum discount? *Ans.* \$2324·84.
7. Find the discount on a bill of \$2202 at 5 per cent., payable 9 months hence. *Ans.* \$79·59036.
8. What is the present worth of a note for \$4360, payable one year 5 months hence, at 6 per cent.? *Ans.* \$4018·43317.
9. What is the present worth of a note for \$1647, due 11 months hence, at 6 per cent.? *Ans.* \$1561·13744.

10. Required the present worth of a note for $2000 due 3 years 7 months hence, at 6 per cent. *Ans.* $1646·09053.

11. What is the discount on a note for $2070·90, payable 1 year 7 months hence, at 5 per cent.? *Ans.* $151·019.

12. What is the present worth of a note of $970·63, payable in 11 months, at 8 per cent.? *Ans.* $904·313.

NOTE.—When the payments are to be made at different times, find the present value of the sums separately; their sum will be the present value of the note, and, as before, this subtracted from the whole amount will give the discount.

13. What is the discount on $3024, the one half payable in 6 and the remainder in 12 months, 7 per cent. per annum being allowed? *Ans.* $150·0464.

14. A merchant owes $440, payable in 20 months, and $896, payable in 24 months; the first he pays in 5 months, and the second in one month after that. What did he pay, allowing 8 per cent. per annum? *Ans.* $1200.

BANK DISCOUNT.

37. Bank Discount is a charge made by a bank for the payment of money on a note before the note is due, and differs materially from discount as commonly calculated.

38. Banks consider the discount to be the same as the interest on the whole amount of the note, from the time it is discounted until the time it becomes due. Bank Discount is therefore greater than the true discount by the interest on the discount.

39. The three days of grace, which by mercantile usage, are allowed to elapse after a note falls due, before it is payable, are always included by banks in the time for which they calculate the discount.

40. Two kinds of notes are discounted at banks:

1st. *Business notes or business paper.* These are notes actually given by one individual to another for property sold or value received.

2nd. Accommodation notes, called also accommodation paper. These are notes made for the purpose of borrowing money from the banks.

41. To find the bank discount on a note:—

RULE.

Add 3 days to the time which the note has to run before it becomes due, and calculate the interest for that time at the given rate per cent.

EXAMPLE.—What is the bank discount on a note of $700, payable in 60 days, allowing discount at 6 per cent.?

OPERATION.

Here the time the note has to run is 72 days = 2 months 12 days.

Interest of $1 at 6 per cent. for 2 months 12 days, is $0·012.
Interest of $700 at 6 per cent. for 2 months 12 days=$0·012 × 700=$8·40. *Ans.*

EXERCISE* 114.

1. What is the bank discount on a note for $986, having 2 years and 8 months to run, allowing discount at 7 per cent.?
Ans. $155·8701.

2. If I have a note for $640, payable in 100 days, and get it discounted at the rate of 8 per cent. per annum, what discount am I charged?
Ans. $14·6488.

3. I sell a horse and carriage for $563·80, and receive a note for that sum, payable, without interest, 91 days hence. Now if I get this discounted at the rate of 6 per cent. per annum, what sum do I receive?
Ans. $554·967.

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42. It is often necessary to make a note of which the present value shall be a certain sum.

Thus, suppose I require to receive from the bank $1000, and wish to give my note, payable in 7 months, at 6 per cent., what amount must I put on the face of the note?

Now the interest on $1 at 6 per cent. for 7 months and 3 days (i. e. days of grace) is $0·0355, and this will be the bank discount on $1 for 7 months at 6 per cent.

To get the present value of $1, we subtract $0·0355 from $1, which gives us $0·9645.

Hence for every $0·9645 I receive, I must put $1 on the face of the note; and therefore to receive $1000, I must put $\frac{1000}{0·9645}$ i. e. $1036·806 on the face of the note.

PROOF.—Face of note.. $1036·806
Bank discount on $1036·806 at 6 per cent. per an. for 7 mon.. 36·806
Present value.. $1000·00

Hence to find the face of a note, due at some future time and discounted at a given rate per cent. per annum, that shall have a known present value, we have the following:—

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* These examples are worked by the rule given in Arts. 26 and 27. If the absolutely correct answer is required, it must be obtained by deducting from these results $\frac{1}{73}$ of the interest for the *days* used, as before explained. In example 2, it will be observed, this makes a difference of 20 cents.

RULE.

Find the present value of $1 for the same time (adding the three days of grace) and at the same rate; divide the required present value of the note by this, and the quotient will be the face of the note.

EXAMPLE.—For what sum must a note be drawn at 8 months 18 days, so that discounted immediately at 6 per cent. it shall produce $670?

OPERATION.

Interest on $1 for 8 months 21 days at 6 per cent.=$0·0435, and this taken from $1 gives us $0·9565=present worth of $1.

Then $\frac{670}{0·9565}$ = $700·47. *Ans.*

EXERCISE* 115.

1. What sum must I put on the face of a note payable in 90 days so that I may obtain $3755 when discounted at a bank at 7 per cent? *Ans.* $3824·15.
2. For what sum must a note be drawn payable in 6 months in order that its proceeds at 5 per cent. bank discount may be $1147·80? *Ans.* $1177·734.
3. For what sum must a note be drawn payable in 45 days so that its proceeds at 3½ per cent. bank discount may be $713·90? *Ans.* $717·2471.

EQUATION OF PAYMENTS.

43. Equation of payments is the process of finding *the equated* or *average time* when two or more payments, due at different times, may be made *at once* without loss to either party.

44. The average time for the payment of several sums due at different times is called the *mean* time or *equated* time.

45. To find the equated time for any number of payments:—

RULE.†

First multiply each debt by the time before it becomes due; then divide the sum of the products thus obtained by the sum of the payments, and the quotient will be the equated time required.

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* Work by Arts. 26 and 27.

† This rule is based upon the supposition that what is gained by keeping certain payments after they become due is equal to what is lost by paying other payments before they become due. This, however, is not exactly true; for the gain is the interest, while the loss is equal only to the

Note.—When there are both days and months, they must all be reduced to the same unit; *i. e.*, the payments must all be reckoned for so many days, or so many months or parts of a month. If one of the payments is due on the day from which the equated time is reckoned, the corresponding product will be nothing; but in finding the *sum* of the debts, this payment must be added with the others. (See Example 3 below.)

Example 1.—A merchant purchases a vessel for $7000, $2000 to be paid in 3 months, $2000 in 5 months, and the balance in 11 months. Now if he wishes to make the whole in one payment for what time must his note be drawn?

OPERATION.

$\$2000 \times 3 = \$6000 \times 1$
$2000 \times 5 = 10000 \times 1$
$3000 \times 11 = 33000 \times 1$

7000) $49000(7 months. *Ans.*

Explanation.—The interest of $2000 for three months is equal to the interest of $6000 for one month. Similarly, the interest of the second payment is equal to the interest of $10000 for one month, and the interest of the third payment is equal to the interest of $33000 for one month. Hence, the interest of the several payments, at the given times, will be equal to that of $49000 for one month; and if we divide this $49000 by the sum of the payments, $7000, we obtain 7 months for the equated time.

That is, $7000 : $49000 :: 1 month : *Ans.* $= \frac{\$49000 \times 1}{\$7000} = 7$ months.

Example 2.—A person owes another £20, payable in 6 months; £50, payable in 8 months; and £90, payable in 12 months. At what time may all be paid together, without loss or gain to either party?

OPERATION.

£	£
$20 \times 6 =$	120
$50 \times 8 =$	400
$90 \times 12 =$	1080
160	160)1600(10 months. *Ans.*
	1600

Example 3.—A debt of $450 is to be paid thus: $100 immediately, $300 in four, and the rest in 6 months. When should it be paid altogether?

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discount, which (Art. 38) is always less than the interest: but the discrepancy is so trifling as not to make any material difference in the result.

With this exception, the rule is true, and may be demonstrated as follows:—Let p = first payment, and t = the time before it becomes due;
p' = other payment, and t' = the time before it becomes due;
x = equated time, and r = the rate of interest per unit.
And since x, the equated time, lies between t and t' the time between t and x is $= x - t$, and that between t' and x is $= t' - x$.
The interest of p for the time $x - t$ is (from Art. 18) $pr(x - t)$.
Also interest of p' for the time $t' - x$ is $p'r(t' - x)$.
Hence $pr(x - t = p'r(t' - x)$.

And $x = \frac{pt + p't'}{p + p'}$, which is the rule, and may be similarly proved for any number of payments.

OPERATION.

$$
\begin{array}{rl}
\$100\times 0= & 0\\
300\times 4= & 1200\\
50\times 6= & 300\\
\hline
450 & 450)1500(3\tfrac{1}{3}\text{ months. } \textit{Ans.}\\
 & 1350\\
\hline
 & \left.\begin{array}{c}150\\ \hline 450\end{array}\right\}=\tfrac{1}{3}
\end{array}
$$

Exercise 116.

1. A owes B $600, of which $200 is payable in 3 months, $150 in 4 months, and the rest in 6 months; but it is agreed that the whole sum shall be paid at one payment. When should the payment be made? *Ans.* In 4½ months.
2. A debt is to be discharged in the following manner: ¼ at present, and ¼ every three months after until all is paid. What is the equated time? *Ans.* 4½ months.
3. A debt of $120 will be due as follows: $50 in 2 months, $40 in 5, and the rest in 7 months. When may the whole be paid together? *Ans.* In 4¼ months.
4. I owe $1000 to be paid down, $1500 in one month, $600 in 3 months, $700 in 5 months, and $1400 in 7 months. For what time must my note be drawn so that the whole may be paid in one payment? *Ans.* $3\frac{5}{26}$ months.
5. Bought of Messrs. Hendrie & Robarts, goods to the following amounts, on the credit of six months:

15th of January,	a bill of $3750.
10th of February,	a bill of 3000.
6th of March,	a bill of 2400.
8th of June,	a bill of 2250.

I wish on 1st of July to give my note for the amount; at what time must it be made payable? *Ans.* 31st August.

PARTNERSHIP OR FELLOWSHIP.

46. Partnership or Fellowship is the joining together of two or more persons for the transaction of business, agreeing to share the profits and losses in proportion to the amount of money each invests in the business.

47. The persons thus associated are called *Partners*, and the association itself a *Company* or *Firm*.

48. The money employed is called the *Capital* or *Stock*.

49. The gain or loss to be shared is called the Dividend.

SIMPLE PARTNERSHIP.

50. When the partners employ their shares of the capital for the same period of time, the partnership is called Simple Partnership.

It is also called Simple Partnership or Partnership without Time.

51. It is evident that the whole stock which suffers the gain or loss must bear the same proportion to the stock of each partner that the whole gain or loss bears to his share of the gain or loss.

Hence, for partnership without time, we have the following :—

RULE.

As the whole stock is to each man's share of the stock, so is the whole gain or loss to each man's share of the gain or loss.

EXAMPLE.—A and B enter into trade with a capital of $3700, of which A contributes $2000 and B the remainder. They gain $1200. What is each man's share of the profits?

OPERATION.

Whole stock : A's stock : : whole profit : A's profit.

That is, \$3700 : \$2000 : : \$1200 : $\frac{2000 \times 1200}{3700}$ = \$648·648 = A's share.

Again, whole stock : B's stock : : whole profit : B's profit.

That is, \$3700 : \$1700 : : 1200 : $\frac{1700 \times 1200}{3700}$ = \$551·351 = B's share.

NOTE.—After A's share has been found, B's share may be obtained by subtracting A's profit from the whole profit.

EXERCISE 117.

1. Two merchants enter into partnership with a stock of $4300, of which A contributes $3000. They gain $1117. How should this be divided between them? *Ans.* A's share = $779·302.
B's share = $337·697.
2. Three persons A, B and C, agree to form a company for the manufacture of woollen cloths. A puts in $6470, B $3780, and C $9860. By the end of the year they find that they have gained $7890. What portion of this profit belongs to each?
Ans. A's share = $2538·458.
B's share = $1483·053.
C's share = $3868·493.
3. B and C buy certain merchandize, amounting to $320, of which B pays $120, and C $200; and they gain $80. How is it to be divided? *Ans.* B $30 and C $50.
4. B and C gain by trade $728; B put in $1200, and C $1600. What is the gain of each? *Ans.* B $312 and C $416.
5. Two persons are to share $100 in the proportions of 2 to B and 1 to C. What is the share of each?
Ans. B $66·66⅔ and C $33·33⅓.

6. A merchant failing, owes to B £500 and C £900; but has only £1100 to meet these demands. How much should each creditor receive? *Ans.* B £392$\frac{6}{7}$ and C £707$\frac{1}{7}$.
7. Three merchants load a ship with butter; B gives 200 casks, C 300, and D 400; but when they are at sea it is found necessary to throw 180 casks overboard. How much of this loss should fall to the share of each merchant?
Ans. B should lose 40 casks, C 60, and D 80.
8. Three persons are to pay a tax of $100, according to their estates. B's yearly property is $800, C's $600, and D's $400. How much is each person's share?
Ans. B's $44·44$\frac{4}{9}$, C's $33·33$\frac{1}{3}$, and D's $22·22$\frac{2}{9}$.
9. Divide 120 into three such parts as shall be to each other as 1, 2, and 3. *Ans.* 20, 40, and 60.
10. A ship worth $900 is entirely lost; $\frac{1}{8}$ of it belonged to B, $\frac{1}{4}$ to C, and the rest to D. What should be the loss of each, $540 being received as insurance? *Ans.* B $45, C $90, and D $225.
11. Three persons have gained $1320; if B were to take $6, C ought to take $4, and D $2. What is each person's share?
Ans. B's $660, C's $440, and D's $220.
12. Three persons join; B and C put in a certain stock, and D puts in £1090; they gain £110, of which B takes £35, and C £29. How much did B and C put in; and D's share of the gain?
Ans. B put in £829 6s. 11$\frac{11}{23}$d.,
C " £687 3s. 5$\frac{17}{23}$d.,
and D's part of the profit is £46.

COMPOUND PARTNERSHIP.

52. When the partners employ their capital for different periods of time, the partnership is called Compound Partnership or Compound Fellowship.

It is likewise called Double Partnership, or Partnership With Time.

For example; suppose A puts in $200 for 8 years, and B $300 for 4 years, and they make a certain gain or loss. This would give a case of Compound Partnership.

In such cases it is plain that each man's share of the profit depends upon two circumstances:

1st. The amount of his stock; and

2nd. The period for which it is continued in the business.

Also that when the times are equal, the shares of the gain or loss are as the stocks; when the stocks are equal, the shares are as the times; and when neither the times nor the stocks are equal, the shares are as their products.

Hence, for Compound Partnership we have the following:—

RULE.

Multiply each man's stock by the time he continues it in trade; then say, as the sum of the products is to each particular product, so is the whole gain or loss to each man's share of the gain or loss.

EXAMPLE.—A contributes $120 for 6 months, B $336 for 11 months, and C $384 for 8 months; and they lose $56. What is C's share of the loss?

OPERATION.

$$\left.\begin{array}{r}\$120\times\ \ 6=\$720 \text{ for one month}\\ 336\times 11=3696 \text{ for one month}\\ 384\times\ \ 8=3072 \text{ for one month}\end{array}\right\}=\$7488 \text{ for one month.}$$

$$\$7488 : \$3072 :: \$56 : \text{C's share; or} \frac{\$3072\times 56}{7488}=\$22{\cdot}974.$$

EXPLANATION.—It is clear that $120 contributed for 6 months are, as far as the gain or loss is concerned, the same as 6 times $120, or $720, contributed for one month. Hence A's contribution may be taken as $720 for 1 month; and, for the same reason, B's as $3696 for the same time; and C's as $2072, also for the same time. This reduces the question to one in Simple Fellowship.

EXERCISE 118.

1. Three merchants enter into partnership; B puts in $357 for 5 months, C $371 for 7 months, and D $154 for 11 months; and they gain $347·20. What should be each person's share of it?
Ans. B's $102, C's $148·40, and D's $96·80.

2. B, C, and D pay $160 as the year's rent of a pasture. B puts 40 cows on it for 6 months, C 30 for 5 months, and D 50 for the rest of the time. How much of the rent should each person pay?
Ans. B $87·27$\frac{3}{11}$, C $54·54$\frac{6}{11}$, and D $18·18$\frac{2}{11}$.

3. Three dealers, A, B, and C, enter into partnership, and in a certain time make £291 13s. 4d. A's stock, £150, was in trade 6 months; B's, £200, 3 months; and C's, £125, 16 months. What is each person's share of the gain?
Ans. A's is £75, B's, £50, and C's, £166 13s. 4d.

4. Three persons have received $665 interest; B had put in $4000 for 12 months, C $3000 for 15 months, and D $5000 for 8 months. How much is each person's part of the interest?
Ans. B's $240, C's $225, and D's $200.

5. Three troops of horse rent a field, for which they pay $320; the first sent into it 56 horses for 12 days, the second 64 for 15 days, and the third 80 for 18 days. What must each pay?
Ans. The first must pay $ 70,
The second " 100,
The third " 150.

6. Three merchants are concerned in a steam-vessel; the first, A, puts in $960 for 6 months; the second, B, a sum unknown for 12 months; and the third, C, $640, for a time not known when the accounts were settled. A received $1200 for his stock and profit, B $2400 for his, and C $1040 for his: what was B's stock, and C's time?
Ans. B's stock was $1600; and C's time was 15 months.

NOTE.—If A gain \$240 in 6 months, he would gain \$480 in 12 months; that is, A's stock and profit at the end of 12 months would be \$960 + \$480 = \$1440.

Then \$1440 : 2400 : : \$960 : B's stock; or $\frac{2400 \times 960}{1440}$ = \$1600 B's stock.

Again, B's stock : C's stock : : B's profit : C's profit for same time, viz.: 12 months. That is \$1600 : \$640 : : \$800 : $\frac{640 \times 800}{1600}$ = \$320 = C's profit for 12 months.

Lastly, C's profit for 12 months : C's given profit : : 12 months : C's time; that is, \$320 : \$400 : : 12 months : $\frac{400 \times 12}{320}$ = 15 months, C's time.

7. In the foregoing question A's gain was \$240 during 6 months, B's \$800 during 12 months, and C's \$400 during 15 months; and the sum of the products of their stocks and times is 34560. What were their stocks?
Ans. A's was \$ 960,
B's " 1600,
C's " 640.

8. In the same question the sum of the stocks is \$3200; A's stock was in trade 6 months, B's 12 months, and C's 15 months; and at the settling of accounts, A is paid \$240 of the gain, B \$800, and C \$400. What was each person s stock?
Ans. A's was \$960, B's \$1600, and C's \$640.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers following the questions refer to the articles of the Section.*

1. What is interest? (1)
2. What is the meaning of the terms *per cent.* and *per annum?* (1)
3. In what respect does interest differ from Commission and Brokerage? (2)
4. What is the principal? (3)
5. What is meant by the *rate per cent?* (4)
6. What is meant by the *rate per unit?* (5)
7. What is the interest? (6)
8. What is the amount? (7)
9. Of how many kinds is interest? (8)
10. Explain the distinction between Simple and Compound Interest. (9 and 10)
11. In using formulas for interest, what is the meaning of the letters *P. A. I, t,* and *r?* (12)
12. Deduce algebraically a full set of rules for Simple Interest. (12)
13. How is the interest found when the *principal, rate per cent.*, and *time* are given? (13)

    NOTE.—Answer this and succeeding similar questions by giving the formula.
14. Interpret this formula. (13)
15. When the *interest, rate per cent.*, and *time* are given, what is the rule for finding the principal? (14)
16. Interpret this formula. (14)
17. How is the rate per cent. found when the *interest, principal*, and *time* are given? (15)
18. Interpret this formula. (15)
19. When the *interest, principal*, and *rate* are given, how is the time found? (16)

20. Interpret this formula. (16)
21. When the *principal*, *rate*, and *time* are given, how is the amount found? (17)
22. Interpret this formula. (17)
23. When the *amount*, *rate*, and *time* are given, how do we find the principal? (18)
24. Interpret this formula. (18)
25. When the *amount*, *principal*, and *time* are given, how do we find the rate? (19)
26. Interpret this formula. (19)
27. When the *amount*, *principal*, and *rate* are given, how do we find the time? (20)
28. Interpret this formula. (20)
29. How do we find the time in which any sum of money will amount to any given number of times itself at a given rate? (21)
30. Interpret this formula. (21)
31. How do we find the rate at which any sum will amount to a given number of times itself in a given time? (22)
32. Interpret this formula. (22)
33. When the time and rate are given, how do we find to how many times itself a given sum will amount? (23)
34. Interpret this formula. (23)
35. How do we find the interest on $1 at 6 per cent. per annum for any number of months? (24)
36. How do we find the interest on $1 at 6 per cent. for any number of days? (25)
37. How do we find the interest of any sum for any given time at 6 per cent.? (26)
38. How may we find the interest at any other rate than 6 per cent.? (27)
39. How do we compute interest on notes, &c., when partial payments are made? (28)
40. What is the rule for calculating Compound Interest? (30)
41. How is Compound Interest calculated by the table given in Art. 31? (32)
42. How do we ascertain the present worth of a debt due some given time hence, allowing Compound Interest at a given rate? (33)
43. What is Discount? (34)
44. What is meant by the present worth of a debt, note, &c.? (35)
45. How do we compute the present worth of a debt or note? (36)
46. What is Bank Discount? (37)
47. What is the distinction between Bank Discount and True Discount? (38 and 35)
48. What are days of grace? (39)
49. What are the two kinds of notes discounted at banks? (40)
50. How do we calculate the bank discount on notes, &c.? (41)
51. How do we find what amount to put on the face of a note so that its present value shall be a certain sum? (42)
52. What is meant by the Equation of Payments? (43)
53. What is meant by the *mean time* or *equated time* of payment? (44)
54. How do we find the equated time of payment? (45)
55. What is Partnership or Fellowship? (46)
56. What are the persons associated together in partnership called? (47)
57. What is the money employed in the business called? (48)
58. What is meant by the dividend? (49)
59. What is the distinction between Simple and Compound Fellowship? (50 and 52)
60. By what other name is Simple Partnership known? (50)
61. What is the rule for Simple Partnership? (51)
62. What is the rule for Compound Partnership? (52)

SECTION IX.

PROFIT AND LOSS, BARTER, ALLIGATION, CURRENCIES, EXCHANGE, &c.

PROFIT AND LOSS.

1. Profit and loss is a rule by which we are enabled to ascertain what we gain or lose in mercantile transactions. It also instructs us how much we must increase or diminish the price of our goods in order that our gain or loss may be so much per cent.

CASE I.

2. *To find the total gain or loss* on a certain quantity of goods when the prime cost and selling price are given:

FIRST RULE.

Find the price of the goods at prime cost and also at the selling price. The difference will be the whole gain or loss.

Example 1.—What do I gain if I buy 207 cords of wood at $3·78 per cord and sell it at $4·25?

OPERATION.

207 cords @ $4·25 = $879·75 = whole sum for which goods were sold.
207 cords @ $3·78 = $782·46 = whole cost.

Difference = $97·29 = whole gain = *Ans.*

Example 2.—If I purchase 900 bushels of wheat at $1·47 per bushel and sell it at $1·25, what do I lose upon the whole transaction?

OPERATION.

900 bushels @ $1·47 = $1323 = whole cost.
900 bushels @ $1·35 = $1125 = whole sum received for wheat.

$198 = whole loss = *Ans.*

SECOND RULE.

Find the difference between the buying and selling price of a bushel, lb., yard, &c.

Multiply the gain or loss per bushel, lb., yard, &c., by the number of bushels, lbs., or yards, and the result will be the whole gain or loss.

EXAMPLE.—Bought 211 yards of flannel at 37½ cents per yard, and sold it at 45 cents. Required my total gain?

OPERATION.

$0·375 = buying price.
$0·45 = selling price.

$0·075 = gain per yard $0·075 × 211 = $15·825. *Ans.*

NOTE.—This second rule affords the shorter method of finding the gain or loss.

EXERCISE 119.

1. Bought 317 lbs. of butter at 9 cents per lb., and sold it at 12½ cents. What was my gain on the whole? *Ans.* $11·095.
2. Bought 2138 bushels of potatoes at 87½ cents per bushel, and sold them at $1·20. What was my gain on the whole? *Ans.* $694·85.
3. Bought 13 barrels of sugar, each weighing 317 lbs. net at 15 cents per lb., and sold the whole for $735. How much did I gain or lose on the transaction? *Ans.* Gained $116·85.
4. Bought 17 kegs of wine, each containing 22 gallons, at $3·15 per gallon, and paid in addition $26 33 for carriage, &c., and an *ad valorem* duty of 37½ per cent. I sold the whole for $1625. What was my gain or loss? *Ans.* Loss $21·2175.

CASE II.

3. Let it be required to find for what sum I must sell a house which cost $2900 so that I may gain 15 per cent.

Here for every $100 the house cost me I am to receive $115, or for every $1 cost I am to receive $1·15.

The selling price must evidently be as many times $1·15 as the buying price contains $1; i. e., $1·15 × 2900 = $3335·00. *Ans.*

Again: If a person buys a horse for $230, and afterwards sells it so as to lose 11 per cent.; how much does he receive for it?

Here for every $1 he paid for the horse he receives only $0·89 (since he loses 11 per cent., i. e. 11 cents on the $1.)

Then, the selling price will obviously be $0·89 × 230 = $204·70. *Ans.*

Hence, *to find at what price* an article must be sold so as to gain or lose a specified per centage, the cost price being given:—

RULE.

Find (Art. 2, Sect. VII.) *how much must be received for each dollar of the buying price, and multiply this by the whole buying price. The result will be the selling price.*

EXAMPLE 1.—Bought a quantity of oatmeal for $1793·80. For what must I sell it so as to gain 8 per cent.?

OPERATION.

Here for every $1 I expend I desire to receive $1·08; hence, the selling price will be $1·08 × 1793·80 = $1937·304. *Ans.*

Example 2.—Bought a lot of sheep for $7000, and am willing to lose 3 per cent. For what sum must I sell?

OPERATION.

Here for every $1 I expend I am willing to receive $0·97, and hence selling price will be $0·97 × 7000 = $6790. *Ans.*

Exercise 120.

1. Bought cordwood at $3·25 per cord. At what rate per cord must I sell it in order to gain 30 per cent? *Ans.* $4·22½.
2. Bought a stock of goods for $13420. For how much must it be sold in order to gain 5 per cent? *Ans.* $14091.
3. Bought a quantity of wool at 11 cents a lb., and wish to sell so as to gain 15 per cent. At what rate per lb. must I sell it? *Ans.* 12½$\frac{3}{20}$ cents.
4. Bought axes at $15·25 a doz., and desire to sell them so as to gain 23 per cent. At what rate per doz. must I sell? *Ans.* $18·75¾.
5. Bought a farm for $7890, and am willing to lose 11 per cent. At what price must I sell? *Ans.* $7022·10.

CASE III.

4. Let it be required to find what per cent. of profit a merchant makes by buying tea at 43 cents per lb. and selling it at 67 cents.

Here the gain on each lb. is 24 cents.

That is every 43 cents invested gives a gain of 24 cents.

Therefore every *cent* invested gains $\frac{1}{43}$ of 24 cents $= \frac{24}{43}$ cents.

And hence, the gain per cent. $= \frac{24}{43} \times 100 = \frac{2400}{43} = 55{\cdot}8$ per cent.

Hence *to find the rate per cent. of profit or loss* when the prime cost and selling price are given, we have the following:—

RULE.

Find the difference between the buying and selling price, and hence the gain or loss per unit.

Multiply this by 100, *and the result will be the gain or loss per cent.*

Example.—A speculator invests $44400 in stocks, and sells out for $50000. What per cent. does he make by the operation?

OPERATION.

Here the whole gain is $50000 − $44400 = $5600.

That is $44400 gain $5600, and therefore $1 gains $\frac{5600}{44400} = \frac{14}{111}$ of a dollar.

Hence gain per cent. $= \frac{14}{111} \times 100 = \frac{1400}{111} = 12{\cdot}6$. *Ans.*

Note.—The above and all similar questions may be solved by Proportion. Thus this question is, if $44400 gain $5600, what will $100 gain?

And the statement is $44400 : $100 :: $5600 : *Ans.* $= \frac{5600 \times 100}{44400} = 12{\cdot}6$.

EXERCISE 121.

1. Bought tea at 60 cents a lb., and sold it at $87\frac{1}{2}$ a lb.; how much did I gain per cent.? *Ans.* $45\frac{5}{6}$.
2. Bought coffee at 13 cents and sold it at 11 cents a pound; what was my loss per cent.? *Ans.* $15\frac{5}{13}$.
3. Bought flour at $6 20 a barrel, and sold it at $7·80; what was the per cent. of profit? *Ans.* $25\frac{1}{8}$ per cent.
4. Bought cloth at $2·75 per yard, and sold it at $3·10; what was my gain per cent.? *Ans.* $12\frac{8}{11}$ per cent.
5. Bought oats at $0·47 per bushel, and sold them at $0·56; what was my gain per cent.? *Ans.* $19\frac{7}{47}$ per cent.
6. Bought meat at 12 cents per lb., and sold it at $10\frac{1}{2}$ cents a pound; what was my loss per cent.? *Ans.* $12\frac{1}{2}$ per cent.
7. Bought a horse for $93, and sold it for $127; what per cent. of profit did I make? *Ans.* $36\frac{52}{93}$.
8. A man bought a farm for $6742·50, and sold it for $6000; what was his loss per cent? *Ans.* $11\frac{11}{899}$ per cent.
9. If I purchase a house for $5700, a horse for $275, and pay $1987·32 for household furniture and a carriage, and then sell the whole for $8750, what is my gain or loss per cent.? *Ans.* Gain 9·89 or nearly 10 per cent.
10. I purchase 723 y[illegible] velvet in Paris and pay $4·25 a yard; I further pay 7 per cent. for insurance, $23·70 for carriage, $2·70 for harbor dues, $3·16 for wharfage and storage, and an *ad valorem* duty of 22 per cent., and then sell the whole for $5270; what is my gain or loss per cent.? *Ans.* Gain 31·96749 or nearly 32 per cent.

CASE IV.

5. Let it be required to find the prime cost of cloth which I sold for $4 and gained 10 per cent. thereby.

Here the gain on $1 was 10 cents, or what I sold for $1·10 cost me only $1.
Therefore the cost price will contain $1 as many times as the selling price contains $1·10.
That is the cost price $=\frac{4}{1\cdot10}=$ $3·636. *Ans.*

Hence, *to find the cost price*, the selling price and the gain or loss per cent. being given, we have the following:—

RULE.

Find the gain or loss per unit, and add it to unity if it be gain, but subtract it from unity if it be loss.

Divide the selling price by the quantity thus obtained, and the result will be the cost price.

Or say as $100 + gain per cent. (or as $100 — loss per cent.) is to $100 so is the selling price to the cost price.

Example.—Sold a quantity of coal for $719, and lost 7 per cent. by the transaction; what was the prime cost?

OPERATION.

1st Rule.—Loss on $1 is [illegible] cents, or for every $1 paid I receive $0·93. Hence cost $= \$\frac{719}{.93} = \$778·118$.

2nd Rule.—$93 : $100 :: $719 : *Ans.* $= \frac{100 \times 719}{93} = \$778·118$.

Exercise 122.

1. For what did I buy a quantity of sugar which I [illegible] $24·60, losing 4 per cent.? [illegible] $25·625.
2. A gentleman sold his library for $2360, which [illegible] per cent. less than cost; what did he give for it? *Ans.* $2622·22.
3. A farmer sold his farm for $7400, gaining 1[illegible] per cent. on the prime cost; what did he give for it? *Ans.* $6666·666.
[illegible] A merchant sold a quantity of silk velvet for $3789·40, gaining 17 per cent. by the transaction; required the buying price? *Ans.* $3238·803.
5. Sold a lot of ca[illegible] for $2740, losing 13 per cent. by the transaction; what did I give for them? *Ans.* $3149·425.

BARTER.

6. Barter signifies an exchange of goods or articles of commerce at prices agreed upon so that neither party in the transaction may sustain loss.

7. *The principle of solution depends upon finding the value of the commodity whose price and quantity are given, and thence the equivalent quantity of a second commodity of a given price, or the equivalent price of a given quantity of a second commodity.*

Example 1.—How much tea at $1·10 per lb. ought to be given for 712 lb. of sugar at 13 cents per lb.?

OPERATION.

712 lbs. of sugar at 13 cents per lb. = $92·56, and $92·56 ÷ $1·10 = 84·1454 lbs. = 84 lbs. 2⅓ oz. *Ans.*

Example 2.—I desire to barter 96 lbs. of sugar, which cost me 8 cents per lb., but which I sell at 13 cents, giving 9 months' credit, for calico which another merchant sells for 17 cents per yard, giving 6 months' credit. How much calico ought I to receive?

OPERATION.

I first find at what price I could sell my sugar, were I to give the same credit as he does—

If 9 months give me 5 cents profit, what ought 6 months to give?

$$9 : 6 :: 5 : \frac{6 \times 5}{9} = \frac{30}{9} = 3\tfrac{1}{3} \text{ cents.}$$

Hence, were I to give 6 months' credit, I should charge $8 + 3\frac{1}{3} = 11\frac{1}{3}$ cents per lb. Next—

As my selling price is to my buying price, so ought his selling to be to his buying price, both giving the same credit.

$$11\tfrac{1}{2} : 8 :: 17 : \frac{8 \times 17}{11\tfrac{1}{2}} = 12 \text{ cents.}$$

The price of my sugar, therefore, is 96 × 8 cents, or $7·68; and of his calico, 12 cents per yard.

Hence $\frac{\$7{\cdot}68}{12} = 64$, is the required number of yards.

Exercise 123.

1. A has coffee which he barters at 10 cents the lb. more than it cost him, against tea which stands B in $2, but which he rates at $2·50 per lb. How much did the coffee cost at first? *Ans.* 40 cents.
2. A has silk which cost $2·80 per lb.; B has cloth at $2·50, which cost only $2 the yard. How much must A charge for his silk, to make his profit equal to that of B? *Ans.* $3·50.
3. I have cloth at 8 cents the yard, and in barter charge for it 13 cents, and give 9 months' time for payment; another merchant has goods which cost him 12 cents per lb., and with which he gives 6 months' time for payment. How high must he charge his goods to make an equal barter? *Ans.* At 17 cents.
4. K and L barter. K has cloth worth $1·60 the yard, which he barters at $1·85 with L, for linen cloth at 60 cents per yard, which is worth only 55 cents. Who has the advantage; and how much linen does L give to K for 70 yards of his cloth? *Ans.* L gives K $215\frac{5}{6}$ yards, and K has the advantage.
5. B has five tons of butter, at $102 per ton, and 10½ tons of tallow, at $135 per ton, which he barters with C; agreeing to receive $600·30 in ready money, and the rest in beef at $4·20 per barrel. How many barrels is he to receive? *Ans.* 316.

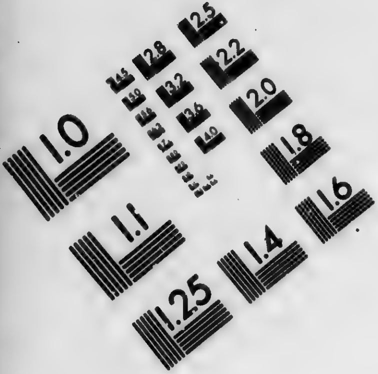
ALLIGATION.

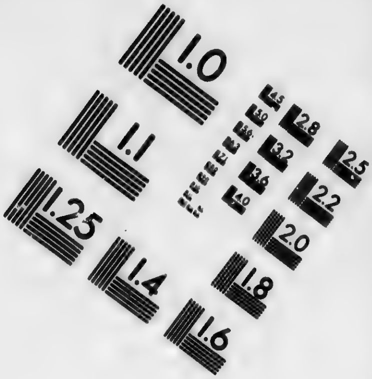
8. Alligation is the method of finding the value of a mixture of ingredients of different values, or of forming a compound which shall have a given value.

Note.—The term *alligation* is derived from the Latin word *alligo* "to tie or bind," the reference being to the manner of connecting or tying the numbers together in a certain class of questions.

9. Alligation is divided into *Alligation Medial* and *Alligation Alternate.*

10. Alligation Medial (Latin *medius*, "mean or average,") enables us to find the value of a mixture when the





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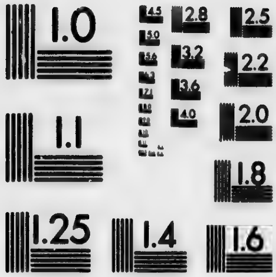

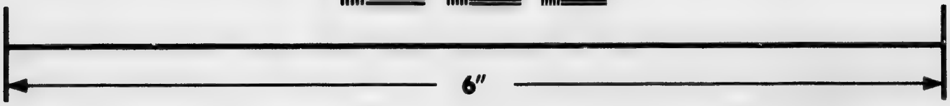

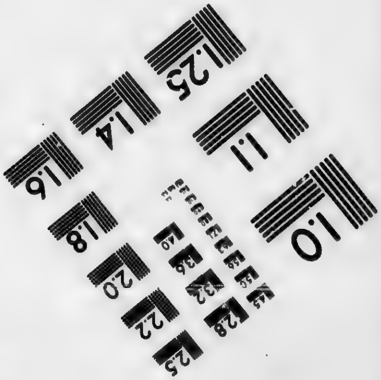

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ingredients, of which it is composed and their prices are known.

11. Alligation Alternate enables us to find what proportion must be taken of several ingredients, whose prices are known, in order to form a compound of a given price.

ALLIGATION MEDIAL.

12. Let it be required to find the price per lb. of a mixture containing 47 lbs. of sugar at 11 cents per lb., 29 lbs. at 13 cents, and 24 lbs. at 17 cents.

OPERATION.

47 lbs. at 11 cents=517 cents.
29 lbs. at 13 cents=377 cents.
24 lbs. at 17 cents=408 cents.

Then 100 lbs. cost 1302 cents and 1 lb. will cost $\frac{1302}{100}$ = 13$\frac{1}{50}$ cents.

Hence for Alligation Medial we deduce the following—

RULE.

Divide the entire cost of the whole mixture by the sum of the ingredients, and the quotient will be the price per unit of the mixture.

EXAMPLE 1.—What will be the price per lb. of a mixture of tea containing 7 lbs. at $0·50 per lb., 11 lbs. at $0·80, 19 at $1·06, and 3 lbs. at $1·23?

OPERATION.

7 lbs. @ $0·50 = $3·50
11 " @ $0·80 = $8·80
19 " @ $1·06 = $20·14
3 " @ $1·23 = $3·69

40 lbs.=sum of ingredients. $36·13=Total cost.
40)$36·13($0·90$\frac{13}{40}$. *Ans.*
360
·13

EXAMPLE 2.—A goldsmith has 3 lbs. of gold 22 carats fine, and 2 lbs. 21 carats fine. What will be the fineness of the mixture?

In this case the value of each kind of ingredient is represented by a number of *carats*—

OPERATION.

3 lbs. × 22=66 carats.
2 " × 21=42 "
5 5)108 "

The mixture is 21$\frac{3}{5}$ carats fine.

EXERCISE 124.

1. Having melted together 7 oz. of gold 22 carats fine, $12\frac{1}{2}$ oz. 21 carats fine, and 17 oz. 9 carats fine, I wish to know the fineness of each ounce of the mixture? *Ans.* $15\frac{44}{73}$ carats.
2. A vintner mixed 2 gallons of wine, at 14s. per gallon, with 1 gallon at 12s., 2 gallons at 9s., and 4 gallons at 8s. What is one gallon of the mixture worth? *Ans.* 10s.
3. A farmer mixes 15 bushels of wheat worth $1·20 with 30 bushels worth $1·50, and 60 bushels worth $1·10, and 83 bushels worth $1·75. What is one bushel of the mixture worth? *Ans.* $1·458.
4. A grocer mixes together 12 lbs. of tea at 50 cents, 16 lbs. at 72 cents, 12 lbs. at 65 cents, 18 lbs. at 85 cents, and 100 lbs. at 42 cents. How much per lb. is the mixture worth? *Ans.* $53\frac{1}{71}$ cents.

ALLIGATION ALTERNATE.

13. Alligation Alternate is the reverse of Alligation Medial, and may be proved by it.

CASE I.

14. Given the prices of the ingredients, to find the proportion in which they must be mixed in order that the compound may be worth a given price:—

RULE.

Set down the prices of the ingredients in two columns, placing those greater than the price of the compound to the left, and those less than it to the right.

Between these columns form two others composed of the differences between the prices of the several ingredients and of the compound; writing each difference next to the number by which it was obtained.

Link, by means of a line, the left-hand differences to the right-hand differences in any order.

Then each difference will express how much of the quantity with whose difference it is connected, should be taken to form the required mixture.

If any difference is connected with more than one other difference, it is to be considered as repeated for each of the differences with which it is connected; and the sum of the differences with which it is connected is to be taken as the required amount of the quantity whose difference it is.

EXAMPLE 1.—How many pounds of tea at 5s. and 8s. per lb., would form a mixture worth 7s. per lb.?

OPERATION.

$$\begin{array}{ccc} \text{Prices.} & \text{Differences.} & \text{Prices.} \\ 7 = 8 - & \overbrace{1 \text{———————} 2} & + 5 = 7 \end{array}$$

1 is connected with 2s., the difference between the 7, the required price, and 5s.: hence there must be 1 lb. at 5s. 2 is connected with 1, the difference between 8s. and the required price; hence there must be 2 lbs. at 8s. Then 1 lb. of tea at 5s. and 2 lbs. at 8s. per lb., will form a mixture worth 7s. per lb.—as may be proved by the last rule.

It is evident that any equimultiples of these quantities would answer equally as well; hence a great number of answers may be given to such a question.

Example 2.—How much sugar at 9d., 7d., 5d., and 10d., will produce sugar at 8d. per lb.?

OPERATION.

$$\begin{array}{ccc} \text{Prices.} & \text{Differences.} & \text{Prices.} \\ 8 = \left\{ \begin{array}{l} 9 - \\ 10 - \end{array} \right. & \overbrace{\begin{array}{c} 1 \text{———————} 1 \\ 2 \text{———————} 3 \end{array}} & \left. \begin{array}{l} + 7 \\ + 5 \end{array} \right\} = 8 \end{array}$$

1 is connected with 1, the difference between 7d. and the mean, 8; hence there is to be 1 lb. of sugar at 7d. per lb. 2 is connected with 3, the difference between 5d. and the mean; hence there is to be 2 lbs. at 5d. 1 is connected with 1, the difference between 9d. and the mean; hence there is to be 1 lb. at 9d. And 3 is connected with 2, the difference between 10d. and the mean; hence there are to be 3 lbs. at 10d per lb

Consequently we are to take 1 lb. at 7d. and 2 lbs. at 5d., 1 lb. at 9d. and 3 lbs. at 10d. If we examine the price of the mixture these will give (Art. 12) we shall find it to be the given mean.

Example 3.—What quantities of tea at 4s., 6s., 8s., and 9s. per lb., will produce a mixture worth 5s.?

OPERATION.

$$\begin{array}{ccc} \text{Prices.} & \text{Differences.} & \text{Prices.} \\ 5 = \left\{ \begin{array}{l} 8 - \\ 6 - \\ 9 - \end{array} \right. & \overbrace{\begin{array}{c} 3 \text{———————} 1 \\ 1 \text{———————} \\ 4 \text{———————} \end{array}} & \begin{array}{l} + 4 = 5 \\ \\ \\ \end{array} \end{array}$$

3, 1, and 4 are connected with 1s., the difference between 4s. and the mean; therefore we are to take 3 lbs. + 1 lb. + 4 lbs. of tea, at 4s. per lb. 1 is connected with 3s., 1s., and 4s., the differences between 8s., 6s., and 9s., and the mean; therefore we are to take 1 lb. of tea at 8s., 1 lb. of tea at 6s., and 1 lb. at 9s. per lb.

Example 4.—How much of any thing at 3s., 4s., 5s., 7s., 8s., 9s., 11s., and 12s. per lb., would form a mixture worth 6s. per lb.?

OPERATION.

$$\begin{array}{ccc} \text{Prices.} & \text{Differences.} & \text{Prices.} \\ 6 = \left\{ \begin{array}{r} 7 - \\ 8 - \\ 9 - \\ 11 - \\ 12 - \end{array} \right. & \overbrace{\begin{array}{c} 1 \text{———————} 3 \\ 2 \text{———————} 2 \\ 3 \text{———————} 1 \\ 5 \text{———————} \\ 6 \text{———————} \end{array}} & \left. \begin{array}{l} + 3 \\ + 4 \\ + 5 \\ \\ \\ \end{array} \right\} = 6 \end{array}$$

1 lb. at 3s., 2 lbs. at 4s., 3 lbs. at 7s., 2 lbs. at 8s., 3+5+6; i. e., 14 lbs. at 5s., 1 lb. at 9s., 1 lb. at 11s., and 1 lb. at 12s. per lb. will form the required mixture.

Note.—The principle upon which this rule proceeds is that the excess of one ingredient above the mean is made to counterbalance what the other wants of being equal to the mean. Thus in example 7, 1 lb. at 5s. per lb. gives a *deficiency* of 2s.; but this is corrected by 2s. *excess* in the 2 lbs. at 8s. per lb.

In example 8, 1 lb. at 7d. gives a *deficiency* of 1d., 1 lb. at 9d. gives an *excess* of 1d.; but the excess of 1d. and the deficiency of 1d. exactly neutralize each other.

Again, it is evident that 2 lbs. at 5d. and 3 lbs. at 10d. are worth just as much as 5 lbs. at 8d.—that is, 8d. will be the average price if we mix 2 lbs. at 5d. with 3 lbs. at 10d.

Exercise 125.

1. How much wheat at $1·60, $1·40, $1·10, and $1 per bushel must be mixed together in order to form a mixture, worth $1·25 per bushel? Give at least *two* sets of answers.

Ans. 35 bushels at $1·10, 15 at $1·60, 15 at $1·00, and 25 at $1·40.
35 bushels at $1·00, 15 at $1·40, 15 at $1·10, and 25 at $1·60.

2. How much wine at 60 cents, 50 cents, 42 cents, 38 cents, and 30 cents per quart, will make a mixture worth 45 cents a quart? *Ans.* 15 qts. at 42 c., 5 qts. at 30 c., 3 qts. at 60 c., and 22 qts. at 50 c., and 5 quarts at 38 cents.

3. A merchant has sugar worth 10 cents, 12 cents, 14 cents, 15 cents, 16 cents, 17 cents, and 18 cents per pound, and wishes to form a mixture worth 12½ cents a lb. How many pounds of each must he use. *Ans.* 2½ lbs. at 14 c., 1½ lbs. at 10 c., 16 lbs. at 12 c., and ½ lb. at each other price.

4. A grocer has sugar at 5d., 7d., 12d., and 13d. per lb. How much of each kind will form a mixture worth 10d. per lb.?

Ans. 2 lbs. at 5d., 3 lbs. at 7d., 5 lbs. at 12d., and 3 lbs. at 13d.

CASE II.

15. When a given quantity of one of the ingredients is to be taken:—

I. Find the proportional quantities of the ingredients as in Case I.

II. Then say, as the amount of the ingredient as thus found is to the given amount of the same ingredient, so is the amount of any other ingredient (found by Case I.) to the required quantity of that other.

Example 1.—29 lbs. of tea at 4s. per lb. is to be mixed with teas at 6s., 8s., and 9s. per lb., so as to produce what will be worth 5s. per lb. What quantities must be used?

OPERATION.

By Case I we find that 8 lbs. of tea at 4s., and 1 lb. at 6s., 1 lb. at 8s., and 1 lb. at 9s., will make a mixture worth 5s. per lb.

Therefore 8 lbs. (the quantity of tea at 4s. per lb., as found by the rule): 29 lbs. (the given quantity of the same tea) : : 1 lb. (the quantity of tea at 6s. per lb., as found by the rule;) or $\frac{1 \times 29}{8}$ lb. = 3⅝ lbs. *Ans.*

We may in the same manner find what quantities of tea at 8s. and 9s. per lb., correspond with 29 lb. of tea at 4s. per lb.

EXAMPLE 2.—A refiner has 10 oz. of gold 20 carats fine, and melts it with 16 oz 18 carats fine. What must be added to make the mixture 22 carats fine?

10 oz. of 20 carats fine = 10 × 20 = 200 carats.
16 oz. of 18 carats fine = 16 × 18 = 288

26 : 1 :: 488 : $18\frac{10}{13}$ carats, the fineness of the mixture.

24 − 22 = 2 carats baser metal in a mixture 22 carats fine.
24 − $18\frac{10}{13}$ = $5\frac{3}{13}$ carats baser metal in a mixture $18\frac{10}{13}$ carats fine.

Then 2 carats : 22 carats :: $5\frac{3}{13}$: $57\frac{7}{13}$ carats of pure gold—required to change $5\frac{3}{13}$ carats baser metal into a mixture 22 carats fine. But there are already in the mixture $18\frac{10}{13}$ carats gold: therefore $57\frac{7}{13} - 18\frac{10}{13} = 38\frac{10}{13}$ carats gold are to be added to every ounce. There are 26 oz.; therefore $26 \times 38\frac{10}{13} = 1008$ carats of gold are wanting. There are 24 carats in every oz.; therefore $\frac{1008}{24}$ carats = 42 oz. of gold must be added. There will then be a mixture containing:—

oz.	car.	car.
10 ×	20 =	200
16 ×	18 =	288
42 ×	24 =	1008

68 : 1 oz. :: 1496 : 22 carats, the required fineness.

EXERCISE 126.

1. How much molasses at 16 cents, at 19 cents, and at 23 cents per quart must be mixed with 87 quarts at 31 cents in order that the mixture may be worth 25 cents per quart?
Ans. $30\frac{12}{17}$ qts. at each price.

2. How much oats at 37 cents per bushel and barley at 68 cents per bushel must be mixed with 70 bushels of peas at 80 cents a bushel so that the mixture may be worth 75 cents per bushel?
Ans. $7\frac{7}{9}$ bush. at each price.

3. How much brass at 14d. per lb., and pewter at $10\frac{1}{2}$d per lb., must I melt with 50 lbs. of copper at 16d. per lb., so as to make the mixture worth 1s. per lb.?
Ans. 50 lbs. of brass, and 200 lbs. of pewter.

4. How much gold of 21 and 23 carats fine must be mixed with 30 oz. of 20 carats fine, so that the mixture may be 22 carats fine?
Ans. 30 of 21, and 90 of 23.

CASE III.

16. When the quantity of the compound is given as well as the price:—

I. Find the proportional quantities as in Case I.

II. Then say, as the sum of the proportional quantities is to each proportional quantity, so is the given quantity to the corresponding part of each.

EXAMPLE.—What must be the amount of tea at 4s. per lb. in 736 lb. of a mixture worth 5s. per lb., and containing tea at 6s., 8s., and 9s., per lb.?

To produce a mixture worth 5s. per lb., we require 8 lbs. at 4s., 1 at 8s., 1 at 6s., and 1 at 9s. per lb. (Art. 14.) But all of these added together, will make 11 lbs. in which there are 8 lbs. at 4s. Therefore

$$\begin{array}{ccccc} \text{lbs.} & \text{lbs.} & \text{lbs.} & \text{lbs.} & \text{lbs. oz.} \\ 11 & : 8 :: & 736 : & \frac{8\times 736}{11} & = 585\ 4\tfrac{4}{11}, \end{array}$$

the required quantity of tea at 4s.

That is, in 736 lbs. of the mixture there will be 585 lbs. $4\frac{4}{11}$ oz. at 4s. per lb. The amount of each of the other ingredients may be found in the same way.

EXERCISE 127.

1. A druggist is desirous of producing, from medicine at $1·00, $1·20, $1·60, and $1·80 per lb., 168 lbs. of a mixture worth $1·40 per lb.; how much of each kind must he use for the purpose? *Ans.* 28lbs. at $1·00, 56lbs. at $1·20, 56lbs. at $1·60, and 28lbs. at $1·80 per lb.
2. 27 lbs. of a mixture worth 4s. 4d. per lb. are required. It is to contain tea at 5s. and at 3s. 6d. per lb.; how much of each must be used? *Ans.* 15lbs. at 5s., and 12lbs. at 3s. 6d.
3. How much brandy at $2·40, $2·60, $2·80, and $2·90, per gallon, must there be in one hogshead of a mixture worth $2·70 per gallon? *Ans.* 18 gals. at $2·40, 9 gallons at $2·60, $9 gals. at $2·80, and 27 gals. at $2·90 per gallon.

EXCHANGE OF CURRENCIES.

17. Exchange of Currencies is the process of changing a sum of money expressed in the denomination of one country to an equivalent sum expressed in the denominations of another country.

18. By *the currency of a country* is meant the *coins,* or *money,* or *circulating medium* of trade of that country.

19. The *intrinsic value* of a coin is determined by the *kind, purity,* and *quantity,* of metal it contains.

20. The *relative value* or *commercial value* of a coin is its market value, and is fixed by law and commercial usage.

FOREIGN MONEYS OF ACCOUNT,

WITH THE PAR VALUE OF THE UNIT, AS FIXED BY COMMERCIAL USAGE, EXPRESSED IN DOLLARS AND CENTS.

Austria.—60 kreutzers=1 florin (silver)=	$0·485
Belgium.—100 cents=1 guilder or florin; 1 guilder (silver)=	·40
Brazil.—1000 rees=1 milree=	·828
Bremen.—5 schwares=1 grote; 72 grotes=1 rix-dollar (silver)=	·787
British India.—12 pice=1 anna; 16 annas=1 Company's* rupee=	·445
Buenos Ayres.—8 rials=1 dollar currency (variable), mean value=	·98
Canton.—10 cash †=1 candarines; 10 cand.=1 mace; 10 mace=1 tael=	1·48
Cape of Good Hope.—6 stivers=1 schiling; 8 schilings=1 rix-dollar=	·818
Ceylon.—4 pice=1 fanam; 12 fanams=1 rix-dollar=	·40
Cuba, Colombia and Chili.—8 rials.=1 dollar.=	1·00
Denmark.—12 pfenning=1 skilling; 16 skillings=1 marc; 6 marcs=1 rix-dollar.=	·52
England.—4 farthings=1 penny; 12 pence=1 shilling; 20 shil.=£1=	4·867
France.—10 centimes=1 decime; 10 decimes=1 franc=	·186
Greece.—100 lepta=1 drachme; 1 drachme (silver)=	·166
Holland.—100 cents=1 florin or guilder; 1 florin (silver)=	·40
Hamburgh.—12 pfenning=1 schiling; 16 schil.=1 marc; 3 marcs=1 rix-dollar=	·84
Malta.—20 grains=1 taro; 12 tari=1 scudo; 2½ scudi=1 pezza=	1·00
Milan.—12 denari=1 soldo; 20 soldi=1 lira=	·20
Mexico.—8 rials=1 dollar=	1·00
Monte Video.—100 centesimos=1 rial; 8 rials=1 dollar=	·833
Naples.—10 grani=1 carlino; 10 carlini=1 ducat (silver)=	·80
Norway.—120 skillings=1 rix-dollar specie (silver)=	1·06
Papal States.—10 bajocchi=1 paolo; 10 paoli=1 scudo or crown=	1·00
Peru.—8 rials=1 dollar (silver)=	1·00
Portugal.—400 rees=1 cruzado; 1000 rees=1 milree or crown=	1·12
Prussia.—12 pfennings=1 grosch (silver); 30 groschen=1 thaler or dollar	·69
Russia.—100 copecks=1 ruble (silver)=	·78
Sardinia.—100 centesimi=1 lira=	·186
Sweden.—48 skillings=1 rix-dollar specie=	1·06
Sicily.—20 grani=1 taro; 30 tari=1 oncia (gold)=	2·40
Spain.—34 maravedis=1 real of old plate ‡=	·10
8 reals=1 piastre; 4 piastres=1 pistole of exchange.	
20 reals vellon=1 Spanish dollar=	1·00

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* The current silver rupee of Bombay, Madras, and Bengal, is worth $0·444. In India also they use *cowries* for coin. These are small shells found in the Maldives and elsewhere: 2560 cowries make a *rupee*, and 100000 *rupees* make a *lac*.

† The cash, made of copper and lead, is said to be the only money coined in China.

‡ The old plate real is not a coin, but is the denomination in which exchanges are usually made.

St. Domingo.—100 centimes = 1 dollar =	$0·923
Tuscany.—12 denari di pezza = 1 soldi di pezza; 2 soldi di pezza = 1 pezza of 8 rials; 1 pezza (silver) =	·90
Turkey.—3 aspers = 1 para; 40 paras = 1 piastre (variable) about	·096
Venice.—100 centesimi = 1 lira =	·186
United States of America.—10 mills = 1 cent; 10 cents = 1 dime; 10 dimes = 1 dollar =	1·00

21. *The following table exhibits the commercial value of the Foreign coins most frequently met with.*

Guinea	$5·10
Sovereign of Great Britain	4·867
Crown of England	1·216
Half-Crown of England	·608
Shilling of England	·24¼
Dollar of the United States	1·00
Franc of France	·18¾
Five-Franc Piece of France	·93
Livre Tournois of France	·18½
Forty-Franc Piece of France	7·66
Crown of France	1·06
Louis-D'Or of France	4·56
Florin of the Netherlands	·40
Guilder of the Netherlands	·40
Florin of Southern Germany	·40
Thaler or **Rix-Dollar** of Prussia and Northern Germany	·69
Rix-Dollar of Bremen	·78¾
Florin of Prussia	·22¼
Marc-Banco of Hamburgh	·35
Florin of Austria and city of Augsburg	[illegible]
Florin of Saxony, Bohemia, and Trieste	·48
Florin of Nuremburg, Frankfort, and Creveld	·40
Rix-Dollar of Denmark	1·00
Specie-Dollar of Denmark	1·05
Dollar of Sweden and Norway	1·06
Milree of Portugal	1·12
Milree of Madeira	1·00
Milree of Azores	·83¼
Real-Vellon of Spain	·05
Real-Plate of Spain	·10
Pistole of Spain	3·97
Rial of Spain	·12
Pistareen	·18
Cross Pistareen	·16
Ruble (silver) of Russia	·75
Imperial of Russia	7·83

Doubloon of Mexico	$15·60
Half-Joe of Portugal	8·58
Lira of Tuscany and Lombardy	·16
Lira of Sardinia	·18⅗
Ounce of Sicily	2.40
Ducat of Naples	·80
Crown of Tuscany	1·05
Florence **Livre**	·15
Genoa "	·18¾
Geneva "	·21
Leghorn **Dollar**	·90
Swiss **Livre**	·27
Scudo of Malta	·40
Turkish **Piastre**	·05
Pagoda of India	1·84
Rupee of India	.44½
Tael of China	1·48

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22. In Canada all accounts were kept in pounds, shillings, pence, and farthings, previous to the adoption of the decimal coinage by Act of Provincial Parliament in 1858. In the United States also accounts were similarly kept prior to the adoption of Federal Money in 1786. In the States, at the time Federal Money was adopted, the *Colonial currency* or *bills of credit* had become more or less depreciated in value, i. e., a colonial shilling was worth less than a shilling sterling, &c., and the depreciation in value being greater in the currencies of some colonies than in others gave rise to the *different values* of the present old currencies of the different States.

TABLE OF CURRENCIES

IN CANADA AND THE UNITED STATES.

In Canada, Nova Scotia, New Brunswick, &c.,	$1 = 5s.	or £$\frac{1}{4}$.
In N. Y., N. C., Ohio, and Mich.,	$1 = 8s.	or £$\frac{2}{5}$.
In N. Eng., Va., Ky., Ten., Ia., Ill., Miss., and Missouri,	$1 = 6s.	or £$\frac{3}{10}$.
In Penn., New Jer., Del., and Md.,	$1 = 7s. 6d.	or £$\frac{3}{8}$.
In Georgia and S. C.,	$1 = 4s. 8d.	or £$\frac{7}{30}$.

Note.—The remaining States use the Federal money exclusively.

23. To reduce dollars and cents to old Canadian Currency, or to any State Currency:—

RULE.

Multiply the given sum by the value of $1 in the required currency expressed as a fraction of a pound. The product will be pounds and decimals of a pound.

Reduce (Art. 58, Sect. IV.) decimals to shillings, pence, and farthings.

Example 1.—Reduce $493·72 to Old Canadian Currency.

OPERATION.

493·72 × ¼ = £123·43 = £123 8s. 7⅕d. *Ans.*

Example 2.—Reduce $749·80 to New England Currency.

OPERATION.

749·80 × $\frac{3}{10}$ = £224·94 = £224 18s. 9⅗d. *Ans.*

Example 3.—Reduce $1111·11 to New York Currency.

OPERATION.

1111·11 × ⅖ = £444·444 = £444 8s. 10$\frac{14}{25}$d. *Ans.*

Exercise 128.

1. Reduce $1974·80 to New Jersey Currency. *Ans.* £740 11s.
2. Reduce $765·43 to Michigan Currency. *Ans.* £306 3s. 5$\frac{7}{25}$d.
3. Reduce $8172·19 to Old Canadian Currency. *Ans.* £2043 0s. 11⅖d.

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24. To Reduce Old Canadian Currency or any State Currency to dollars and cents:—

RULE.

Express the given sum decimally and divide it by the value of a dollar expressed as a fraction of a pound; the quotient will be dollars, cents, &c.

Example 1.—Reduce £179 18s. 4¾d., Old Canadian Currency, to dollars and cents.

OPERATION.

£179 18s. 4¾d. = £179·9197916 and 179·9197916 ÷ ¼ = $719·67916. *Ans.*

Note.—Old Canadian Currency may be most expeditiously reduced to dollars and cents by the rule given in Art. 89, Sect. I.

Example 2.—Reduce £234 18s. 9¼d., Ohio Currency, to dollars and cents.

OPERATION.

£234 18s. 9¼d = £234·9885416 and 234·9885416 ÷ ⅖ = $587·84635416. *Ans.*

Exercise 129.

1. Reduce £743 18s. 11d., New England Currency, to dollars and cents. *Ans.* $2479·8194.
2. Reduce £119 9s. 8¼d., Maryland Currency, to dollars and cents. *Ans.* $318·625.
3. Reduce £473 17s. 1¾d., Georgia Currency, to dollars and cents. *Ans.* $2030·816964.

25. To reduce dollars and cents to sterling money :—

RULE.

Divide the given sum by the value of £1 sterling ($4·8674), *the quotient will be pounds sterling and decimals of a pound.*

Reduce the decimal part (Art. 58, Sect. IV) *to shillings and pence.*

EXAMPLE.—Reduce $749·88 to sterling money.

OPERATION.

749·88 ÷ 4·867 = £154·0641 = £154 1s. 3½d. *Ans.*

EXERCISE 180.

1. Reduce $1006·90 to sterling money. *Ans.* £206 17s. 7½d.
2. Reduce $916·87 to sterling money. *Ans.* £188 7s. 8½d.
3. Reduce $2114·81 to sterling money. *Ans.* £434 10s. 4½d.

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26. To reduce sterling money to dollars and cents :—

RULE.

Express the given sum decimally and multiply by the legal value of £1 sterling ($4·867).

EXAMPLE.—Reduce £78 11s. 4¾d. to dollars and cents.

OPERATION.

£78 11s. 4¾d. = £78·5697916 and 78·5697916 × 4·867 = $382·399. *Ans.*

EXERCISE 181.

1. Reduce £2043 11s. 3d. sterling to dollars and cents. *Ans.* $9946·01868.
2. Reduce £777 7s. 7d. sterling to dollars and cents. *Ans.* $3783·50437.
3. Reduce £557 19s. 5½d. sterling to dollars and cents. *Ans.* $2715·65418.

EXCHANGE.

27. Exchange is a commercial term, denoting the payment of money by a person residing in one place to a person residing in another, by draft or bill of exchange.

28. A bill of exchange is a written order addressed to a person directing him to pay, at a specified time and place, a certain sum of money to another person or his order.

29. The person who signs the bill of exchange is called *the drawer* or *maker* of the bill.

30. The person on whom it is drawn is called the *drawee*, and, after he has accepted it, the *acceptor*.

31. The person to whom the money is directed to be paid is called the *payee*.

32. The person who purchases the bill of exchange, i. e., the person in whose favor it is drawn, is called the *buyer* or *remitter*.

33. The person who has legal possession of the bill is called the *holder*.

34. The *acceptance* of a bill or draft is a promise on the part of the drawee to pay it at maturity or the specified time. The usual mode of accepting a bill is for the drawee to attach his signature to the word "*accepted*," written either across the face of the note or on its back.

NOTE.—A draft or bill of exchange should be presented to the drawee, for his acceptance, immediately on its receipt.

35. If the payee or holder of a bill or draft wishes to sell it or transfer it, he endorses it, i. e., he writes his name on the back.

NOTE.—If the endorser directs the bill to be paid to a particular person, the endorsement is called a *special endorsement* and the person therein named is called the *endorsee*.

If the endorser simply writes his name on the back of the bill, the endorsement is called a *blank endorsement*.

When the endorsement is blank, or when the bill is made payable to bearer, it may be transferred from one to another at pleasure, and the *drawee* is bound to pay it to the holder at maturity. If the drawee or acceptor of a bill fail to pay it, the endorsers are responsible for the payment.

36. When the drawee of a bill refuses acceptance, or, having accepted, fails to make payment when it becomes due, the bill is immediately *protested*.

37. A *protest* is a formal declaration in writing, made by a public officer called a *Notary Public*, at the request of the holders of the bill, notifying the drawer, endorsers, &c., of its non-acceptance or non-payment.

NOTE. If the drawer and endorsers are not notified within a reasonable time of the non-acceptance or non-payment of the bill, they are not responsible for its payment.

When a bill is protested for non-acceptance, the drawer must pay it immediately, even though the specified time has not arrived.

38. The time specified for the payment of a bill varies, and is a matter of agreement between the drawer and buyer. Some are payable at sight, some at a certain number of *days* or *months* after sight or after date. In both cases it is customary to allow *three days of grace*.

39. Bills of Exchange are divided into *inland* and *foreign bills*. When both *drawer* and *drawee* reside in the same country, they are called *inland bills or drafts*; when in different countries, *foreign bills*.

NOTE.—Three bills are commonly drawn for the same amount, &c., and are called respectively the *First*, *Second*, and *Third of Exchange*, and together constitute a set. These are sent by different ships or conveyances; and when the *first* that arrives is accepted or paid, the others become void. This plan is adopted in order to avoid the delays which might arise from accidents, miscarriages, &c.

FORM OF AN INLAND BILL OR DRAFT.

$3000. Toronto, 1*st July*, 1859.

Ten days after sight, pay to the order of George McCallum, Esq., Three Thousand Dollars, value received, and charge the same to

Ridout & Steven.

Messrs. Hardman & Morris,
Bankers, Hamilton.

FORM OF A FOREIGN BILL OF EXCHANGE.

Exchange 8000 francs. Toronto, 17*th July*, 1859.

At sixty days sight of this first of exchange (the second and third of the same date and tenor unpaid) pay to Edward Atkinson, Esq., or order, the sum of Eight Thousand Francs, with or without further advice.

John Henderson.

Messrs. Duhamel & Beauharnois,
Bankers, Paris.

40. The *par of exchange* is that amount of the money of one country actually equal to a given sum of the money of another, and is either *intrinsic* or *commercial.*

41. The *intrinsic par of exchange* is the *real value* of the money of different countries, as determined by the weight and purity of their standard coins.

Thus, the English sovereign is intrinsically worth $4·861 of the gold coin of the United States.

42. The *commercial par of exchange* is a comparison of the coins of different countries, according to their nominal or market value.

Thus, the English sovereign varies in market value from $4·83 to $4·85.

Note.—The intrinsic par is always the same so long as the standard coins are of the same kind, quantity, and quality of metal; the commercial par is determined by commercial usage, and fluctuates, being different at different times.

43. The *Course of Exchange* signifies the *current price* paid in one country for bills of exchange drawn on another.

Note.—The course of exchange is constantly fluctuating from various causes When the exports of a country just equal its imports, the exchange will be at par; when the balance of trade is against a place, i. e., when its imports exceed its exports, bills on foreign countries will be *above* par, because there will be a greater demand for them to pay the bills due abroad; when the balance of trade is in favor of a country, i. e , when its exports exceed its imports, bills of exchange on foreign countries will be *below* par since fewer of them will be required.

The course of exchange can never very greatly exceed the *intrinsic par value*, because when the premium on bills of exchange becomes great it is less expensive to importers to pay for the insurance and transportation of bullion and coin to meet their payments than to transmit bills of exchange.

44. By an old act of Provincial Parliament it was enacted that £100 sterlings or 100 sovereigns should be equivalent to £111⅑ Canadian money, i. e. to $\$444{\cdot}44\dot{4}$ or £1 sterling $= \$4{\cdot}44\dot{4}$. It was found however that this was very much below the real or intrinsic value of the sterling pound, accordingly, while its legal value was only $\$4{\cdot}44\dot{4}$, the market or commercial value varied from \$4·83 to \$4·86. By an act recently passed by the Provincial Parliament the value of the pound sterling was fixed at $\$4{\cdot}86\dot{6}$.

Now the new par is equal to the old par *plus* nine and a-half per cent. of the old par, that is, $\$4{\cdot}44\dot{4}$ + 9½ per cent. of $\$4{\cdot}44\dot{4}$, which is, ·422, make $\$4{\cdot}86\dot{6} =$ the new par. Consequently the rate of exchange between Canada and Great Britain must reach the nominal premium or 9½ per cent. before it is at par, according to the new standard.

45. Rates of exchange between Canada and Great Britain are commonly reckoned, at a certain per cent. on the old par of exchange, instead of on the new par.

EXAMPLE 1.—A merchant in Hamilton wishes to remit to London £749 3s. 6d. sterling; exchange being at 10 per cent. premium. How much must he pay for the bill of exchange?

OPERATION.

Old commercial par of £1 sterling $= \$4{\cdot}44\dot{4}$
To which add 10 per cent. of itself $= {\cdot}44\dot{4}$

Gives price of £1 $= 4{\cdot}88\dot{8}$
Then £749 3s. 6d. = £749·175 × $4{\cdot}88\dot{8}$ = \$3662·63⅓. *Ans.*

EXAMPLE 2.—A merchant in Toronto wishes to remit 144479 francs to Paris, exchange being at a premium of 2 per cent. What will be the cost of his bill in dollars and cents?

OPERATION.

Commercial value of the franc = 18·6 cents.
Add 2 per cent. = ·372 "

Gives value for remitting = 18·972 "
Then 18·972 × 144479 = \$27410·55588. *Ans.*

EXAMPLE 3.—What sum in dollars and cents will purchase a bill of exchange on Hamburg for 14667 marcs banco, exchange being at 1½ per cent. discount?

OPERATION.

Commercial value of the marc banco = 35 cents.
Deduct 1½ per cent. = ·525 "

Gives value for remitting = 34·475 "
Then 34·475 cents × 14667 = \$5056·448. *Ans.*

EXERCISE 132.

1. If I wish to remit $16785·25 to Paris, for how many francs and centimes can I obtain a bill—exchange being 5 francs 4 centimes to the dollar? *Ans.* 84597 francs 66 centimes.
2. What is the cost of a bill of exchange for 4000 marcs banco at one per cent. above par? *Ans.* $1414.
3. How much must I give for a draft on New York for $35678 at $2\frac{1}{4}$ per cent. premium? *Ans.* $36480·755.
4. What will a bill of exchange on St. Petersburg for 2560 rubles cost in dollars and cents, at 2 per cent. discount, the par being 75 cents per ruble? *Ans.* $1881·60.
5. What will be the cost of a bill of exchange on Great Britain for £800 sterling, at 8 per cent. premium? *Ans.* $3840·00

ARBITRATION OF EXCHANGE.

46. Arbitration of exchange is the process of changing a given amount of the money of one country into an equivalent sum of the money of another, through the medium of one or more intervening currencies with which the first and last are compared.

NOTE.—Arbitration enables a person to ascertain whether it is more advantageous to draw or remit a bill of exchange direct from one country to another or indirectly through other places.

47. When there is but *one* intervening country, the operation is termed *simple arbitration;* when there are *two or more* intervening countries, *compound arbitration.*

48. All questions in arbitration of exchange may be solved by one or more statements in simple proportion; it is more convenient, however, to consider them as problems in Conjoined Proportion, and work them by the rule given in Art. 50, Sec. V.

NOTE.—*Care must be taken to reduce all the money of the same country to the same denomination before linking them as directed in the rule.*

EXAMPLE 1.—A merchant in Toronto wishes to remit 2000 marcs banco to Hamburg, and the exchange between Toronto and Hamburg is 35 cents for one marc banco. He finds, however, that the exchange between Toronto and Lisbon is $1·08 for 1 milree, that between Lisbon and Paris is 6 milrees for 38 francs, and that between Paris and Hamburg is 19 francs for 10 marcs banco. How much will he gain by the circuitous exchange?

OPERATION.

$$\begin{array}{lll}
\text{STATEMENT.} & & \text{SAME CANCELLED.}\\
108 \text{ cents} = 1 \text{ milree.} & & {}^{3}108 = 1\\
6 \text{ milrees} = 38 \text{ francs.} & & \cancel{6} = \cancel{38}^{2}\\
19 \text{ francs} = 10 \text{ marcs banco.} & & \cancel{19} = \cancel{10}\\
2000 \text{ marcs banco} = x & & {}^{200}\cancel{2000} = x.
\end{array}$$

$$x = 200 \times 3 \times 108 = \$648.$$

$2000 \times 35 = \$700{\cdot}00$ = what he has to pay by direct exchange.
$648{\cdot}00$ = what he has to pay by circuitous exchange.

Difference = \$ 52·00 = what he gains by the latter mode.

EXAMPLE 2.—£824 Flemish being due to me at Amsterdam, it is remitted to France at 16d. Flemish per franc; from France to Venice at 300 francs per 60 ducats; from Venice to Hamburg at 100d. per ducat; from Hamburg to Lisbon at 50d. per 400 rees; from Lisbon to England at 5s. 8d. sterling per milree; and from England to Canada at \$4·867 per £1 sterling. Shall I gain or lose, and how much, the exchange between Canada and Amsterdam being 7s. 1d. Flemish per dollar?

OPERATION.

$$\begin{array}{lll}
\text{STATEMENT.} & & \text{SAME CANCELLED.}\\
 & & \quad 2\\
16\text{d. Flemish} = 1 \text{ franc.} & & \cancel{16} = 1\\
300 \text{ francs} = 60 \text{ ducats.} & & {}^{50}\cancel{300} = \cancel{60}\\
1 \text{ ducat} = 100\text{d. Flemish.} & & 1 = \cancel{100}\\
50\text{d. Flemish} = 400 \text{ rees.} & & \cancel{50} = \cancel{400}^{8}\\
1000 \text{ rees} = 68\text{d. British.} & & \cancel{10}\ \cancel{4}\ \cancel{1000} = \cancel{68}\ 17\\
240\text{d. British} = \$4{\cdot}867. & & \cancel{240} = 4{\cdot}867 \qquad 3296\\
x = 197760\text{d. Flemish.} & & x = \cancel{197760} \quad \cancel{19776}
\end{array}$$

$$x = \frac{17 \times 4{\cdot}867 \times 3296}{2 \times 50} = \$2727{\cdot}07\tfrac{1}{2} = \text{amount remitted.}$$

Then since exchange between Canada and Amsterdam is 7s. 1d. Flemish per dollar we have

$$\begin{aligned}
85\text{d. Flemish} &= 100 \text{ cents.}\\
x \quad \text{"} \quad &= 197760\text{d. Flemish.}
\end{aligned}$$

Here $x = \frac{197760 \times 100}{85} = \$2326{\cdot}58$ = sum I should have received had it been transmitted direct from Amsterdam to Canada.

Hence by the circuitous exchange I gain the difference between \$2727·07½ and \$2326·58 that is \$400·49½.

EXERCISE 133.

1. If London would remit £1000 sterling to Spain, the direct exchange being 42½d. per piastre of 272 maravedis; it is asked whether it will be more profitable to remit directly, or to remit first to Holland at 35s. per pound; thence to France at 19⅓d. per franc; thence to Venice at 300 francs per 60 ducats; and thence to Spain at 360 maravedis per ducat?

Ans. The circular exchange is more advantageous by 108 piastres, 3 reals, 20 maravedis.

1664185 153600 direct

2. A merchant wishes to remit $4888·40 from Montreal to London, and the exchange is 10 per cent. He finds that he can remit to Paris at 5 francs 15 centimes to the dollar, and to Hamburg at 35 cents per marc banco. Now, the exchange between Paris and London is 25 francs 80 centimes for £1 sterling, and between Hamburg and London 13¾ marcs banco for £1 sterling. How had he better remit?

Ans. If he remits direct to London he will obtain a bill for £1000.

If he remits through Paris he will obtain a bill for only £975 15s. 8½d.

If he remits through Hamburg he will obtain a bill for £1015 15s. 5d.

Hence the best way to remit is through Hamburg, and the next best way is direct to London.

3. A merchant in Quebec wishes to remit 1200 marcs banco to Hamburg, and the exchange of Quebec on Hamburg is 35 cents for 1 marc. He finds the exchange of Quebec on Paris is 18 cents for 1 franc; that of Paris on London, is 25 francs for £1 sterling; that of London on Lisbon, is 180 pence for 3 milrees; that of Lisbon on Hamburg, is 5 milrees for 18 marcs banco. How much will he gain by the circuitous exchange?

Ans. Direct exchange $420; circuitous exchange $375; gain $45.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

Note.—*The numbers after the questions refer to the numbered articles of the section.*

1. What is profit and loss? (1)
2. How do we find the total gain or loss on a quantity of goods when the cost price and selling price are given? (2)
3. How do we find at what price an article must be sold so as to gain or lose a specified percentage, the cost price being given? (3)
4. How do we find the rate per cent. of profit or loss? (4)
5. How do we find the cost price when the selling price and the gain or loss per cent. are given? (5)
6. What is barter? (6)
7. What is alligation? (8)
8. Into what rules is alligation subdivided? (9)
9. What is alligation medial? (10)
10. What is alligation alternate? (11)
11. How is alligation alternate proved? (13)
12. Give the different rules for alligation? (12, 14–16)
13. What is meant by the exchange of currencies? (17)
14. What is meant by the currency of a country? (18)
15. How is the intrinsic value of a coin determined? (19)
16. What fixes the commercial value of a coin? (20)
17. How do you account for the fact that the $ is of different values in the American States? (22)
18. Give the value of the pound currency in Canada, and in the different States. (22)
19. How do we reduce dollars and cents to old Canadian currency or to any state currency? (23)

20. How do we reduce old Canadian Currency or any state currency to dollars and cents? (24)
21. How do we reduce dollars and cents to sterling money? (25)
22. How do we reduce sterling money to dollars and cents? (26)
23. What is a bill of Exchange? (28)
24. Explain the terms *drawer, drawee, acceptor, payee, holder, endorser, and endorsee.* (29–35)
25. How is a bill accepted? (34)
26. What is the difference between a blank endorsement and a special endorsement? (35)
27. What is meant by *protesting* a bill? (36, 37)
28. Explain what is meant by the First, Second, and Third of Exchange. (39)
29. What is the par of Exchange? (40)
30. Explain the difference between the intrinsic par and the commercial par of Exchange. (41, 42)
31. What is the course of Exchange? (43)
32. Explain what is meant by saying the par of Exchange between Canada and Britain is 9½ per cent. (44)
33. Upon what is the rate of Exchange between Canada and Britain reckoned? (45)
34. What is arbitration of Exchange? (46)
35. What is the difference between simple and compound arbitration? (47)
36. By what rule are questions in arbitration of Exchange worked? (48)

SECTION X.

INVOLUTION, EVOLUTION, LOGARITHMS, AND LOGARITHMIC ARITHMETIC.

1. A power of any number is the product obtained by multiplying that number by itself one or more times.

Thus $25 = 5 \times 5$ is a power of 5; $81 = 3 \times 3 \times 3 \times 3$ is a power of 3, &c.

2. The number which, being multiplied once or oftener by itself, produces the power, is called the *root* of that power.

Thus 5 is the root of 25, since $5 \times 5 = 25$; 3 is the root of 81, since $3 \times 3 \times 3 \times 3 = 81$.

3. The powers of a number are called the *first, second, third, fourth, fifth, &c.*, according as the root is taken *once, twice, thrice, four times, five times*, &c., as factor.

Thus 81 is called the fourth power of 3, because 3 is taken 4 times as factor, in order to produce 81.

4. The second power of a number is also called its *square*, because a square surface, the length of one of whose sides is expressed by a given number, will have its area expressed by the second power of that number. (See Art. 62, Sec. I.)

5. The third power of a number is also called its *cube*; because if the length of one side of a cube be expressed by a given number, the solid contents of the cube will be expressed by the third power of that number. (See Art. 64, Sec. I.)

6. The *index* or *exponent* of a power is a small figure written to the right, indicating how often the root has to be taken as factor in order to produce the given power.

Thus $2^1 = 2 \qquad\qquad\qquad\quad = 2$ = First power of 2.
$2^2 = 2 \times 2 \qquad\qquad\quad = 4$ = Second power of 2.
$2^3 = 2 \times 2 \times 2 \qquad\quad = 8$ = Third power of 2.
$2^4 = 2 \times 2 \times 2 \times 2 \quad = 16$ = Fourth power of 2.
$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ = Fifth power of 2.

So also 8^7 means the seventh power of 8; i. e., a number produced by taking 8 seven times as factor, &c.

7. $(5 + 8)^2$ means that the sum of 5 and 8 is to be squared as one number, and is a very different thing from $5^2 + 8^2$, which means the sum of the squares of 5 and 8.

Thus $(5 + 8)^2 = 13^2 = 169$, while $5^2 + 8^2 = 25 + 64 = 89$.

Therefore $(5 + 8)^2 = 25 + 80 + 64 =$ *1st part squared, plus twice product of 1st part by 2nd part, plus 2nd part squared.*

8. The process of finding a power of a given number by multiplying it into itself is called Involution.

9. To involve a number to any required power:—

RULE.

Take the given number as factor as many times as there are units in the index of the required power and find the continued product of these factors.

NOTE.—*Fractions are involved by multiplying both numerators and denominators as above, and mixed numbers should be reduced to fractions before applying the rule.*

EXAMPLE 1.—What is the fifth power of 7?

OPERATION.

Here the index of the required power is 5, and hence the given number 7 must be taken 5 times as factor.

$$7 \times 7 \times 7 \times 7 \times 7 = 16807 \ \textit{Ans.}$$

EXAMPLE 2.—What is the third power of $\frac{3}{4}$?

$$\textit{Ans.}\ (\tfrac{3}{4})^3 = \tfrac{3}{4} \times \tfrac{3}{4} \times \tfrac{3}{4} = \frac{27}{64} \ \textit{Ans.}$$

EXERCISE 134.

1. Find the fifth power of 3.	*Ans.* 243.
2. Required the tenth power of 20.	*Ans.* 10240000000000.
3. Required the sixth power of 1·05.	*Ans.* 1·340095640625.
4. Find the seventh power of $\frac{3}{5}$.	*Ans.* $\frac{2187}{78125}$.
5. Find the fifth power of $\frac{5}{9}$.	*Ans.* $\frac{3125}{59049}$.
6. Required the third power of $11\frac{2}{5}$.	*Ans.* $\frac{185193}{125} = 1481\frac{68}{125}$.

10. Let it be required to find the product of 4^3 by 4^2.

$4^3=4\times4\times4$ and $4^2=4\times4$. Therefore $4^3\times4^2=(4\times4\times4)\times(4\times4)=4\times4\times4\times4\times4$ $=4^5=4^{3+2}$.

Hence two or more powers of the same number are multiplied together by adding their indices or exponents.

Thus, $6^5\times6^2\times6^3=6^{5+2+3}=6^{10}$.
$5\times5^2\times5^3\times5^7=5^{1+2+3+7}=5^{13}$, &c., &c.

11. Let it be required to divide 3^5 by 3^2.

$3^5=3\times3\times3\times3\times3$ and $3^2=3\times3$.

$$\text{Therefore, } 3^5\div3^2=\frac{3^5}{3^2}=\frac{3\times3\times3\times3\times3}{3\times3}=3\times3\times3=3^3=3^{5-2}.$$

Hence, to divide one power of a number by another power of the same number, we subtract the index of the divisor from the index of the dividend.

Thus, $7^6\div7^3=7^{6-3}=7^3$
$3^{11}\div3^4=3^{11-4}=3^7$, &c., &c.

12. Let it be required to find the third power of 7^2.

$$(7^2)^3=7^2\times7^2\times7^2=7\times7\times7\times7\times7\times7=7^6=7^{2\times3}.$$

Hence to find any required power of a given power, we multiply the index of the given power by the index of the required power.

Thus, $(2^4)^5=2^{4\times5}=2^{20}$; $(3^2)^7=3^{2\times7}=3^{14}$, &c., &c.

EXERCISE 135.

1. Multiply together 4^2, 4^4, 4^6, and 4^7. *Ans.* 4^{19}.
2. Divide 13^{11} by 13^2. *Ans.* 13^9.
3. Find the fifth power of 3^3. *Ans.* 3^{15}.
4. Find the value of $\{(7^4\times7^3)\div(7^2\times7^2)\}^6$ *Ans.* 7^{18}.
5. Find the value of $\{(5^3\times5^4\times5^{11}\times5^9)\div(5^3\times5^2\times5^7\times5^5)\}^3$. *Ans.* 5^{30}.

EVOLUTION.

13. Evolution is the process of finding any required root of a given power.

NOTE.—Evolution is the reverse of involution; the latter teaches how to find a power of a number by multiplying it into itself; the former, how to find the root of a power by resolving it into *equal factors.* It follows that powers and roots are correlative terms. If one number is a power of another the latter is a root of the former.

14. A root of a number may be indicated by either of two methods.

1st. By using $\sqrt{}$, called the radical sign (Lat. *radix*, a root).

2nd. By using a fractional index having unity for its numerator, and the number expressing the degree of the root for denominator.

Thus, The square root of 7 is expressed either by $\sqrt{7}$ or by $7^{\frac{1}{2}}$.

The cube root of 6 is " " $\sqrt[3]{6}$ or by $6^{\frac{1}{3}}$.

The seventh root of 2 is " " $\sqrt[7]{2}$ or by $2^{\frac{1}{7}}$.

NOTE.—The figure placed in the radical sign, or as denominator of the fractional index denotes the root.

A fractional index with *numerator* greater than one is sometimes used; in such cases the *denominator* denotes the *root*, and the numerator the power to be taken.

Thus, $2^{\frac{2}{3}}$ means either the cube root of the square of 2 or the square of the cube root of 2.

The radical sign $\sqrt{}$ a corrupted form of the letter *r*, the initial letter of the Latin word *radix*, "a root."

EXERCISE 136.

1. Express the square root of 17 and the cube root of 11.

Ans. $\sqrt{17}$ or $17^{\frac{1}{2}}$ and $\sqrt[3]{11}$ or $11^{\frac{1}{3}}$

2. Express the fifth root of 4. *Ans.* $\sqrt[5]{4}$ or $4^{\frac{1}{5}}$

3. Express the fourth root of 5^3. *Ans.* $\sqrt[4]{5^3}$ or $5^{\frac{3}{4}}$

4. Express the sixth root of 7^4 *Ans.* $\sqrt[6]{7^4}$ or $7^{\frac{4}{6}}=7^{\frac{2}{3}}$

5. Express the third power of the fifth root of 1. *Ans.* $(\sqrt[5]{2})^3$ or $2^{\frac{3}{5}}$

6. Express the eleventh power of the tenth root of 161.

Ans. $(\sqrt[10]{161})^{11}$ or $161^{\frac{11}{10}}$

---

15. Let it be required to extract the fifth root of 3^{15}.

The fifth root of 3^{15} is expressed either by $\sqrt[5]{3^{15}}$, or by $3^{\frac{15}{5}}$.

Taking the latter mode, we have $3^{\frac{15}{5}}=3^3=3^{15\div5}$

Hence, to extract any root of a given power of a number, we divide the index of the power by the index of the root.

Thus, The seventh root of 2^{14} is $2^{14\div7}=2^2$

The fourth root of 2^{12} is $2^{12\div4}=2^3$, &c., &c.

EXTRACTION OF THE SQUARE ROOT.

16. To extract the square root of a number, is to find a number which, being multiplied once by itself, will produce the given number.

RULE.

I. Point off the given number into periods of two figures each, beginning at the decimal point.

II. Find the highest square contained in the left-hand period and place its root to the right of the number, in the place occupied by the quotient in division.

III. Subtract the square of the digit put in the root, from the left-hand period, and to the remainder bring down the next period to the right, for a new dividend.

IV. Double the part of the root already found for a TRIAL DIVISOR.

V. Find how many times the trial divisor is contained in the dividend, exclusive of the right-hand digit, and place the figure thus obtained both in the root and also to the right of the trial divisor.

VI. Multiply the divisor thus completed by the digit last put in the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

VII. Again, double the part of the root already found for a new TRIAL DIVISOR; *proceed as in V. and VI., and continue the process until all the periods are brought down.*

NOTE.—If the given number is not a perfect square, its exact square root cannot be found; but by annexing periods of ciphers, we can obtain any required approximation to it.

EXAMPLE 1.—What is the square root of 22420225?

```
 ˙ ˙ ˙ ˙
22420225(4785, is the required root.
16
 ――
 87)642
    609
    ―――
 943)3302
     2829
     ――――
 9465)47325
      47325
      ―――――
```

EXPLANATION.—Here 22 is the left hand period, and the highest square in 22 is 16, of which the square root is 4. We place 4 in the root and subtract 16 from 22. This leaves a remainder 6, to which we bring down the next period, 42, and thus obtain 642 for the new dividend. Our next step is to find the *trial divisor*, which we obtain by doubling the part of the root already found. This gives us 8, (= 4 doubled) and we ask how many times 8 will go into 64 (the dividend exclusive of the right hand digit). Bearing in mind that we are to put the digit thus obtained both in the root and in the divisor, and that the completed divisor will be over 80, we find that the required digit is 7, which we accordingly place both in the root and in the divisor. The complete divisor is 87, which multiplied by 7, gives 609, and this subtracted from 642, gives a remainder 33, to which we bring down the next period, 02, and thus get 3302 for the next dividend.

Again, doubling the part of the root already found, we obtain 94 (= 47 doubled) for a trial divisor, and as this will go into 330 (the dividend exclusive of the right hand digit) 3 times, we place 3 both in the root and in the divisor. Multiplying the 943 thus obtained by 3, subtracting and bringing down the next period, we get 47325 for the next dividend. The next trial divisor is 946 (= 473 doubled) which will go into 4732 (the dividend exclusive of the right hand figure) 5 times; and we therefore place 5 both in the root and in the divisor. Multiplying and subtracting, we find no remainder; 4785 is therefore the square root of 22420225.

PROOF.—4785 × 4785 = 22420225.

EXPLANATION AND REASON.

17. We may consider every number as consisting of its *tens*, plus its *units*; that is, if the tens be represented by the letter a and the units by the letter b.

Number $= a + b$;
Number squared $= (a + b)^2 = a^2 + 2ab + b^2$.

Hence the square of a number is equal to the square of the tens, plus twice the product of the tens by the units, plus the square of the units.

Thus, $69 = 60 + 9$
And $(69)^2 = (60 + 9)^2 = (60)^2 + 2 \times 60 \times 9 + 9^2 = 3600 + 1080 + 81 = 4761$.

18. Let it now be required to extract the square root of 4761.

I. It is evident that the square of a number consisting of a single digit can never contain more than two digits or less than one; conversely the square root of a number of one or two digits must be a number of one digit. Again the square of a number consisting of two digits can never contain more than four or less than three digits; conversely the square root of a number of three or four digits must be a number consisting of two digits. Similarly, the square of a number consisting of three digits can contain neither more than six nor less than five digits, and conversely, the square root of a number consisting of five or six digits, must be a number of three digits, &c.; that is, one digit in the root is equivalent to two digits in the square, or conversely, two digits in the square are equivalent to one digit in the root.

Hence, if we divide the given number into periods of two figures each beginning at the decimal point, the number of periods will indicate the number of digits in the root.

II. Taking the number 4761, we divide it into periods, thus, 4̇76̇1, and since there are two periods in the square there must be two digits in the root. We thus learn that 4761 is the square of a certain number of tens, plus a certain number of units. Now it is manifest that the square of the tens can only be found in the second period, 47, since tens squared can give no digit of a lower order than hundreds. Also, that no part of the square of the units can be found in the second period, 47, since any single unit squared can give no digit of a higher order than tens.

Therefore the square of the units is found only in the first or lowest period, the square of the tens only in the second period, the square of the hundreds only in the third period, &c.

OPERATION.

4̇76̇1(69 = square root.
36 = highest square in 2nd period.

6 tens × 2 = 12 tens + 9 units = 129) 1161 = remainder which contains, 1st, twice product of tens by units, 2nd, the square of the units
1161 = twice 6 tens $\times 9 + 9^2$.

III. In extracting the square root of this number, we look first for the digit occupying the place of tens in the root. We know (II.) that the square of tens is contained in the second period, 47, and the highest square contained in 47 must be the square of the highest digit that can possibly stand in the place of tens in the root. But the highest square in 47 is 36, the square root of which is 6. Placing 36 under the 47, 6 in the root, we subtract and bring down the next period, 61, and thus get a total remainder of 1161. Now (Art. 17) the whole number 4761 consists of the square of the tens, plus twice the product of the tens by the units, plus the square of the units: and since we have subtracted from it 36, (or if the ciphers be annexed 3600) the square of the tens, the remainder, 1161, must contain twice the product of the tens by the units, plus the square of the units: that is, twice 6 tens × by a certain number of units, plus the square of that number of units, and because we do not know as yet what the units' figure of the root is, we use twice the tens for a trial divisor.

IV. Since we are now seeking the units' digit of the root, and since tens multiplied by units can give no digit of a lower order than tens, the right hand digit of the dividend can form no part of twice the product of the tens by the units, and we have simply to ascertain how often 12 tens (=twice 6 tens) will go in 116 tens. Evidently 9 times.

V. Lastly, we place the digit thus found in the root, because it is a figure of the root, and in the divisor, because the dividend contains not only twice the product of the tens by the units, but also the square of the units. Now when we multiply the 9 by 9 we get the square of the units, and when we multiply the 12 tens by 9 units, we get twice the product of the tens of the root by the units.

Example 2.—Extract the square root of 127449.

OPERATION.

```
    127449(357
    9
    ---
 65)374
    324
    ----
707)4949
    4949
    ----
```

Explanation and Reason.—From the pointing off we learn that the given number is the square of a certain number of hundreds, plus a certain number of tens, plus a certain number of units.

I. We are first then to look for the digit in the place of hundreds, and since hundreds squared can give no digit of a lower order than *tens* of thousands or of a higher order than hundreds of thousands, we see that the square of the hundreds can be found only in the left hand period. The highest square contained in the left hand period is 9, the square root of which is the left hand digit of the entire root.

II. After subtracting, we bring down the next period *only*, because we are now looking for the digit in the place of tens in the root. And since *tens* squared can give no digit of a lower order than *hundreds*, the lowest period cannot enter into any part of the square of tens, much less can it enter into any part of twice the product of the hundreds by the tens, and therefore when looking for the tens of the root, we pay no attention to the right hand period of the square.

III. The remainder of the process is similar, and the reason for the various steps the same as in example 1.

19. To extract the square root of a decimal—

RULE.

I. Annex one cipher, if necessary, in order that the number of decimal places may be even.

II. Point off into periods of two figures each, beginning at the decimal point, and extract the square root as in whole numbers, re-

membering that the number of decimal places in the root will be equal to the number of periods in the square.

EXERCISE 137.

1. Extract the square root of 195864. *Ans.* 442.
2. Extract the square root of ·0676. *Ans.* ·26.
3. Extract the square root of 984064. *Ans.* 992.
4. Extract the square root of 5, true to five decimal places. *Ans.* 2·23606.
5. Extract the square root of ·5, true to six decimal places. *Ans.* ·707106.
6. Extract the square root of 60·487129. *Ans.* 7·777.
7. Extract the square root of 79792266297612001. *Ans.* 282475249.
8. Extract the square root of 0·0000012321. *Ans.* 0·00111.

20. To extract the square root of a fraction—

RULE.

I. Reduce mixed numbers to improper fractions, and compound and complex fractions to simple ones, and the resulting fraction to its lowest terms.

II. Extract the square root of both numerator and denominator separately, if they have exact roots; but if they have not both exact roots, reduce the fraction to its corresponding decimal, by Art. 56, Sec. IV., and then extract the root as in Art. 19.

EXAMPLE 1.—Extract the square root of $2\frac{1}{4}$.

OPERATION.

$$Ans.\ 2\tfrac{1}{4}=\tfrac{9}{4} \text{ and } \sqrt{\tfrac{9}{4}}=\frac{\sqrt{9}}{\sqrt{4}}=\tfrac{3}{2}=1\tfrac{1}{2}.$$

EXAMPLE 2.—Extract the square root of $3\frac{3}{7}$.

OPERATION.

$$3\tfrac{3}{7}=\tfrac{24}{7}=3{\cdot}42857142 \text{ and } \sqrt{3{\cdot}42857142}=1{\cdot}8516.$$

EXERCISE 138.

1. Find the square root of ·1̇. *Ans.* $\frac{1}{3}$.
2. Find the square root of $\frac{1}{121}$. *Ans.* $\frac{1}{11}$.
3. Find the square root of $5\frac{1}{7}$. *Ans.* 2·267786.

Find the square root of $\frac{217}{538}$. *Ans.* ·63509.

5. Find the square root of $13\frac{1}{5}$. *Ans.* 3·63318.

21. Let it be required to extract the square root of 63513·423 accurately.

OPERATION.

```
 63513·4230(236·155 +
 4
43)235
   162
466)4313
    4161
5051)122·42
      50·51
505·25)41·6130
       34·3564
505·335)4·223300
        3·436344
         ·453623
```

EXPLANATION.—We point off into periods of two places each, as in the decimal or common scale. Then the highest square in 6, the first period is 4, of which the square root is 2. Subtracting 4 from the 6 and bringing down the next period, 35, we get 235 for the dividend. Next doubling the 2 we obtain 4, and we find that this will go into 23, the dividend exclusive of the right hand figure, 3 times. Placing this 3 in both root and divisor, multiplying (bearing in mind that 7 is the common ratio of the system) and subtracting, we obtain a remainder of 43, to which we bring down the next period, 13, and thus get 4313 for the next dividend, &c.

EXAMPLE.—Extract the square root of 4731392 *undenary* true to two places to the right of the separating point.

OPERATION.

```
   4731392(2182·99. Ans.
   4
  41)73
     41
 428)3213
     3069
 4358)11592
      8674
 4354·9)3999·00
        2594·t4
4355·79)404·0700
        359·5744
         55·5467
```

EXERCISE 130.

1. Extract the square root of 11333311 *septenary*. *Ans.* 2626.
2. Extract the square root of 33233344 *senary*. *Ans.* 4344.
3. Extract the square root of 4234·10123 *quinary*. *Ans.* 43·412.
4. Extract the square root of 888888·888 *nonary*. *Ans.* 888·88.
5. Extract the square root of 248664et69 *duodenary*. *Ans.* 54373.

APPLICATION OF SQUARE ROOT.

22. A triangle is a figure having three sides, and consequently three angles. When one of the angles is a *right angle*, like the corner of a square, the triangle is called a *right angled triangle*.

23. In a right angled triangle the side *opposite the right angle* is called the *hypothenuse*, and the sides containing the right angle, are called the *base* and the *perpendicular.*

24. It is shown by elementary geometry that the square described on the hypothenuse of a right angled triangle is equal to the sum of the squares described on the other two sides.

Or if h be the hypothenuse, b the base, and p the perpendicular; then

$$h^2 = b^2 + p^2, \text{ and hence}$$
$$h = \sqrt{b^2 + p^2}$$
$$b = \sqrt{h^2 - p^2}$$
$$p = \sqrt{h^2 - b^2}$$

That is—to find the hypothenuse of a right angled triangle when the other sides are given we add the square of the base to the square of the perpendicular and extract the square root of the sum.

To find the length of the base we subtract the square of the perpendicular from the square of the hypothenuse and extract the square root of the remainder.

To find the length of the perpendicular we subtract the square of the base from the square of the hypothenuse and extract the square root of the remainder.

25. The following principles are also established by geometry:—

Circles are to each other as the squares of their diameters.

If the diameter of a circle be multiplied by 3·1416, *the product is the circumference.*

If the square of half the diameter of a circle be multiplied by 3·1416, *the product is the area.*

If the square root of half the square of the diameter of a circle be extracted, it is the side of an inscribed square.

If the area of a circle be divided by 3·1416, *the quotient is the square of half the diameter.*

EXAMPLE 1.—If the hypothenuse of a right angled triangle is 12 feet long and the base 10 feet, how long is the perpendicular?

OPERATION.

$$12^2 = 144$$
$$10^2 = 100$$
$$\text{difference} = 44 \text{ and } \sqrt{44} = 6{\cdot}63324. \ \textit{Ans.}$$

EXAMPLE 2.—If the foot of a ladder be placed 20 feet from the side of a house, how long must it be in order to reach to the top of the house, the latter being 46 feet high?

OPERATION.

$$46^2=2116$$
$$20^2=\ \ 400$$

sum$=2516$ and $\sqrt{2516}=50{\cdot}15$. *Ans.*

Exercise 140.

1. Suppose a ladder 100 feet long be placed 60 feet from the foot of a tree; how far up the tree will the top of the ladder reach? *Ans.* 80 feet.

2. Two persons start from the same place, and go, the one due north 50 miles, the other due west 80 miles. How far apart are they? *Ans.* 94·34 miles, nearly.

3. How large a square stick of timber can be hewn from a round stick 24 inches in diameter? *Ans.* 16·97 in. to the side.

4. A man has a ladder 36 feet long, which, when put on the outside of a ditch 20 feet wide, exactly reaches the top of the wall. Required the height of the wall. *Ans.* 29·933.

5. A ladder 40 feet long is placed against a wall 14 feet high, and just reaches the top; it is then turned over and touches the top of another wall 26 feet high. Required the breadth of the street. *Ans.* 22·622 yds.

6. If the area of a circle be 1760 yards, how many feet must there be in the side of a square to contain that quantity? *Ans.* 125·857.

7. A certain general has an army of 141376 men. How many must he place in rank and file to form them into a square? *Ans.* 376.

8. What is the distance through the opposite corners of a square yard? *Ans.* 4·24264 feet.

9. The distance between the lower ends of two equal rafters, in the different sides of a roof, is 32 feet, and the height of the ridge above the foot of the rafters is 12 feet. What is the length of a rafter? *Ans.* 20 feet.

10. What is the distance measured through the centre of a cube from one corner to its opposite corner, the cube being 3 feet, or 1 yard, on a side? *Ans.* 5·196 feet.

11. If an iron wire $\frac{1}{10}$ inch in diameter will sustain a weight of 450 pounds, what weight might be sustained by a wire an inch in diameter? *Ans.* 45000 lbs.

12. What length of rope must be tied to a horse's neck, in order that he may feed over an acre? *Ans.* 7·136+perches.

13. Four men, A, B, C, D, bought a grindstone, the diameter of which was 4 feet; they agreed that A should grind off his share first, and that each man should have it alternately until he had worn off his share; how much did each man grind off?

Note.—In this question we disregard the thickness of the grindstone. After the first has ground off his portion, there will remain $\frac{3}{4}$ of the stone.

Then the whole stone : part remaining : : square of diameter of whole stone : square of diameter of part remaining. (Art. 25.)

That is, $1 : \frac{3}{4} : : 4^2 : x^2$, and hence $x = 4 \times \sqrt{\frac{3}{4}} = 4 \times \sqrt{\cdot 75} = \cdot 866 \times 4 = 3\cdot 464 =$ diameter of stone after the first has ground off his portion.

Similarly, after the second has ground off his portion there will remain $\frac{1}{2}$ of the stone, and after the third has taken his portion, $\frac{1}{4}$ of the stone.

Hence $1 : \frac{1}{2} : : 4^2 : x^2$, whence $x = 4\sqrt{\frac{1}{2}} = 2\cdot 828$ ft. = diameter after 2nd has taken his portion.

$1 : \frac{1}{4} : : 4^2 : x^2$, whence $x = 4 \times \sqrt{\frac{1}{4}} = 2$ ft. = diameter after 3rd has taken off his portion.

Hence A takes off $4 - 3\cdot 464 = \cdot 536$ ft. $= 6\cdot 432$ inches.
B " $3\cdot 464 - 2\cdot 828 = \cdot 636$ ft. $= 7\cdot 632$ inches.
C " $2\cdot 828 - 2 = \cdot 828$ ft. $= 9\cdot 936$ inches.
D " remaining 2 ft. = 24 inches.

CUBE ROOT.

26. To extract the cube root of a number is to find a number which taken *three times* as factor will produce the given number—

RULE.

I. Point off the number into periods of three figures each beginning at the decimal point.

II. Find the highest cube contained in the left hand period and place its root to the right of the number, in the place occupied by the quotient in division.

III. Subtract the cube of the digit put in the root from the left hand period, and to the remainder bring down the next period to the right for a new dividend.

IV. Multiply the square of the part of the root already found by 300 *for a* TRIAL DIVISOR.

V. Find how many times the trial divisor is contained in the dividend and put the figure thus obtained in the root.

VI. Complete the TRIAL DIVISOR *by adding to it:*

- 1st. *The part of the root previously found × the last digit put in the root × 30 and*
- 2nd. *The square of the last digit put in the root.*

VII. Multiply the divisor thus completed by the digit last put in the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

VIII. Again multiply the square of the part of the root already found by 300 for a new TRIAL DIVISOR, *find what digit to place next in the root as in V, complete the divisor by making the two additions to the third divisor described in VI, multiply, subtract and bring down as directed in VII, and continue the process until all the periods are brought down.*

EXAMPLE.—What is the cube root of 429172932007?

OPERATION.

		429172932007 343	7543 *Ans.*
1st trial divisor $= 7^2 \times 300$	$= 14700$	86172 = 1st dividend.	
1st increment $= 7 \times 5 \times 30$	$= 1050$		
2nd " $= 5^2$	$= 25$		
1st complete divisor	$= 15775$	78875 = product of comp. div by 5.	
2nd trial divisor $= 75^2 \times 300$	$= 1687500$	7297932 = 2nd dividend.	
1st increment $= 75 \times 4 \times 30$	$= 9000$		
2nd " $= 4^2$	$= 16$		
2nd complete divisor	$= 1696516$	6786064 = product of comp. div. by 4.	
3rd trial divisor $= 754^2 \times 300$	$= 170554800$	511868007 = 3rd dividend.	
1st increment $= 754 \times 3 \times 30$	$= 67860$		
2nd " $= 3^2$	$= 9$		
3rd complete divisor	$= 170622669$	511868007 = product of comp. div. by 3.	

EXPLANATION.—After pointing off we find that the highest cube number contained in the left hand period is 343, of which the cube root is 7. We therefore place 7 in the root and subtract 343 from the first period. This gives us a remainder of 86, to which we bring down the next period 172, and thus obtain 86172 for a new dividend.

Next we take 7, the part of the root already found, square it and multiply the 49 thus obtained by 300, this gives the first trial divisor 14700 which we find will go into the dividend 86172 (making due allowance for the increase of the divisor) 5 times.

Next we complete the divisor by adding to it

1st, $7 \times 5 \times 30 = 1050$, and 2nd, $5^2 = 25$ which gives us

15775 for a complete divisor. This we multiply by 5, the digit last put in the root, subtract the product 78875 from the 1st dividend, and to the remainder 7297 bring down the next period 932, &c., &c.

27. EXPLANATION AND REASON.—We have seen (Art. 17) that we may consider every number as consisting of its *tens*, plus its *units*, or if a = tens and b = units, then

$$\text{Number} = a + b; \text{ and}$$
$$\text{Number cubed} = (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Hence the cube of a number is equal to the cube of the tens, plus three times the product of the tens squared multiplied by the units, plus three times the product of the tens multiplied by the square of the units, plus the cube of the units.

$$\text{Thus } 69 = (60+9); \text{ and}$$
$$69^3 = (60+9)^3 = 60^3 + 3 \times 60^2 \times 9 + 3 \times 60 \times 9^2 + 9^3$$
$$= 21600 + 97200 + 14580 + 729$$
$$= 328509.$$

28. Let it now be required to extract the cube root of 328509.

I. It is manifest that the cube of a single digit can never contain more than three digits or less than one digit, and hence the cube root of a number (i. e., perfect cube) of one, two or three digits must be a number of one digit. Again the cube of a number consisting of two digits can never contain more than six or less than four digits, and conversely the cube root of a perfect cube consisting of four, five or six digits must be a number of two digits. Similarly the cube root of a perfect cube consisting of seven, eight or nine digits must be a number of three digits, &c.

Hence, one digit in the root is equivalent to three digits in the cube, and conversely three digits in the cube are equivalent to one digit in the root, and therefore if we divide the given number into periods of three digits each, beginning at the decimal point, the number of periods will indicate the number of digits in the root.

II. The cube of the units can be found only in the period immediately to the left of the decimal point, since any unit cubed can give no digit of a higher order than hundreds. Also the cube of the tens can be found only in the second period to the left of the decimal point, since tens cubed can give no digit of a higher order than *hundreds of thousands*, or a lower order than *thousands*. Similarly the cube of the hundreds can be found only in the third period to the left of the decimal point, &c.

Hence, counting from the decimal point towards the left, the cube of the units can be found only in the first period, the cube of the tens only in the second period, the cube of the hundreds only in the third period, &c.

III. Taking the number 328509 we divide it into periods, thus $\dot{3}28\dot{5}09$, and since there are two periods in the cube there must be two digits in the root.

OPERATION.

$$\begin{array}{rr|l}
 & & \dot{3}28\dot{5}09(69 \\
 & & 216 \\
\cline{3-3}
6^2 = 36 \times 300 = & 10800 & 112509 \\
6 \times 9 = 54 \times 30 = & 1620 & \\
9^2 = & 81 & \\
\cline{2-2}
 & 12501 & 112509
\end{array}$$

We thus learn that 328509 is the cube of a certain number of tens, *plus* a certain number of units. We first then look for the digit in the place of tens in the root. We know (II) that the cube of the tens is contained in the second period 328, and the highest cube contained in 328 must evidently be the cube of the highest digit that can occupy the place of tens in the root—which digit we are seeking. The highest cube contained in 328 is 216, of which the cube root is 6. We then subtract 216 from 328 and to the remainder bring down 509, the next period, which gives us 112509 for a new dividend.

IV. From the given number we have only subtracted 216 (or if the ciphers be affixed, 216000) the remainder, 112509 therefore consists (Art. 27) of three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units; that is, 112509 consists of $(6 \text{ tens})^2 \times 3 \times$ a certain number of units $+ (6 \text{ tens}) \times 3 \times$ (that number of units)$^2 +$ (that number of units)^3; and because we do not know as yet what the units' figure is, we use $(6 \text{ tens})^2 \times 3$ for a trial divisor.

But $(6 \text{ tens})^2 \times 3 = (60)^2 \times 3 = (6 \times 10)^2 \times 3 = 6^2 \times 10^2 \times 3 = 6^2 \times 300$; or in other words, any number of tens squared, multiplied by 3, is equal to that same number of units squared and multiplied by 300. Hence we obtain the constant multiplier 300.

V. $6^2 = 36$, and this multiplied by 300 gives us 10800. In asking how often this is contained in 112509 we have to bear in mind that we must increase the trial divisor by the two additions indicated in the sixth section of the rule. Making allowance for these additions, we find the units' figure of the root to be 9.

VI. If we were to multiply the 10800 we have obtained as a trial divisor by 9, the units' figure of the root, we should only get three times the product of the square of the tens by the units; but we require also three times the product of the tens by the square of the units, and lastly the cube of the units. Our complete divisor must therefore evidently consist of—

1st. Three times the square root of tens.
2nd. Three times the tens multiplied by the units.
3rd. The square of the units; or representing the tens by a and the units by b, the divisor must $=3a^2+3ab+b^2$, and this multiplied by b, the digit in the units' place will give

$$(3a^2+3ab+b^2)b=3a^2b+3ab^2+b^3=\text{the dividend.}$$

Now (6 tens) × 3 = (60) × 3 = 6 × 10 × 3 = 6 × 30, i. e., the product of any number of *tens* multiplied by 3 is equal to the product of that same number of *units* multiplied by 30.

Hence we obtain the constant multiplier 30.

The additions we make then are $6\times30\times9=1620$, and $9^2=81$, and thus we obtain the complete divisor $12501=(60)^2\times3+60\times3\times9+9^2$, and multiplying this by 9, we get

$\left\{(60)^2\times3+60\times3\times9+9^2\right\}9=60^2\times3\times9+60\times3\times9^2+9^3=$ three times the square of the tens multiplied by the units, plus three times the tens multiplied by the square of the units, plus the cube of the units.

NOTE.—When there are more than two periods, the reasons are analogous, since we never have to do with more than *tens* and *units* of the root at one time; i. e., when we are seeking the second digit of the root, we call the first digit tens and the second, units; when we are seeking the third digit of the root, we consider the first two as so many tens, and the third as units, &c.

The reason for bringing down only one period at a time is similar to the reason for the same step in the extraction of the square root (for which see Art. 18, Example 2).

29. To extract the cube root of a decimal—

RULE.

I. Annex two ciphers, if necessary, in order to make the last period complete.

II. Point off into periods of three places each, beginning at the decimal point, and extract the cube root as in whole numbers, remembering that the number of decimal places in the root will be equal to the number of periods in the cube.

EXERCISE 141.

1. What is the cube root of 62712728317? *Ans.* 3973.
2. Extract the cube root of 1953125. *Ans.* 125.
3. Extract the cube root of 1076890625. *Ans.* 1025.
4. What is the cube root of ·697864103? *Ans.* ·887.
5. What is the cube root of 102503·232? *Ans.* 46·8.
6. Find the cube root of 179597·069288. *Ans.* 56·42.
7. Find the cube root of 483·736625. *Ans.* 7·85.
8. Find the cube root of ·636056. *Ans.* ·86.

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30. To extract the cube root of a mixed number or a vulgar fraction—

RULE.

Reduce mixed numbers to improper fractions, and compound or *complex fractions to simple ones, and the resulting fraction to its lowest terms.*

II. Extract the cube root of both numerator and denominator separately, if they have exact roots; but if they have not both exact roots, reduce the fraction to its corresponding decimal by Art. 56, Sect. IV, and then extract the root as in Art. 29.

EXAMPLE 1.—What is the cube root of $3\frac{3}{8}$?

OPERATION.

$$\sqrt[3]{3\tfrac{3}{8}}=\sqrt[3]{\tfrac{27}{8}}=\frac{\sqrt[3]{27}}{\sqrt[3]{8}}=\tfrac{3}{2}=1\tfrac{1}{2}.\ \textit{Ans.}$$

EXAMPLE 2.—Extract the cube root of $17\frac{1}{8}$.

OPERATION.

$$17\tfrac{1}{8}=17{\cdot}125,\ \text{and}\ \sqrt{17{\cdot}125}=2{\cdot}577,\ \text{nearly.}$$

EXERCISE 142.

1. Extract the cube root of $\frac{2}{19}$.	*Ans.* ·4721.
2. Extract the cube root of $\frac{3}{17}$.	*Ans.* ·5609.
3. Extract the cube root of $\frac{1}{3}$ of $2\frac{1}{2}$.	*Ans.* ·941.
4. Extract the cube root of $28\frac{3}{4}$.	*Ans.* 3·063.
5. Extract the cube root of $32\frac{8}{11}$.	*Ans.* 3·198.

31. In extracting the cube root of a number in any scale, other than the decimal, we proceed in the same manner, pointing off into periods of three figures each, finding a trial divisor and afterwards completing it as in the preceding examples.

NOTE.—In all scales having a radix higher than 3, the constant multipliers are 300 and 30; but as in the *binary* and *ternary* scale we cannot use a digit so high as 3, these multipliers become respectively 1100 and 110 for the *binary* scale, and 1000 and 100 for the *ternary* scale.

EXAMPLE 3.—Extract the cube root of 613412·132 *septenary*.

OPERATION.

	$\dot{6}13\dot{4}12{\cdot}13\dot{2}(65{\cdot}04$
	426
$6^2=51\times300=21300$	154412
$6\times30=240\times5=\ \ 1560$	
$5^2=\ \ 34$	
23224	152456
$65^2=6304\times300\ =\ 2521500$	1623·132
$650^2=630400\times300\ =252150000$	1623·132000
$650\times30=26100\times4=\ \ 143400$	
$4^2=\ \ 22$	
252323422	1402·630321
	220·201346

ction to its low-

d denominator
not both exact
al by Art. 56,

Ans. ·4721.
Ans. ·5609.
Ans. ·941.
Ans. 3·063.
Ans. 3·198.

mber in any
he same man-
each, finding
as in the pre-

nstant multipliers
nnot use a digit so
10 for the *binary*

2 *septenary.*

Exercise 143.

1. Express one million in the *senary* scale and then extract its cube root. *Ans.* 244.

Extract the cube root of 6131271 *octenary.* *Ans.* 165·32.

3. Extract the cube root of 10221012·102 *ternary.* *Ans.* 112·012.
4. Extract the cube root of *teteet* in the *duodenary* scale true to two places to the right of the separating point. *Ans.* e7·t2.
5. Extract the cube root of 421030·4412 *quinary* true to two places to the right of the separating point. *Ans.* 44·004.

32. Since many teachers prefer Horner's method of extracting the cube root to the common method, we shall give it here. Upon closely examining it the student will find that the reasons for the several steps of the process are identical with those given in Arts. 27 and 28. The constant multipliers 300 and 30 are still used, but in a disguised form.

RULE.

I. Point off as in the common method.

II. Find the greatest cube in the first period on the left hand; place its root, on the right of the number for the first figure of the root, and also in col. I. on the left of the number. Then multiplying this figure into itself, set the product for the first term in col. II.; and multiplying this term by the same figure again, subtract this product from the period, and to the remainder bring down the next period for a dividend.

III. Adding the figure placed in the root to the first term in col. I., multiply the sum by the same figure, add the product to the first term in col. II., and to this sum annex two ciphers, for a divisor; also add the figure of the root to the second term of col. I.

IV. Find how many times the divisor is contained in the dividend, and place the result in the root, and also on the right of the third term of col. I. Next multiply the third term thus increased by the figure last placed in the root, and add the product to the divisor; then multiply this sum by the same figure, and subtract the product from the dividend. To the remainder bring down the next period for a new dividend.

V. Find a new divisor in the same manner that the last divisor was found, then divide, &c., as before; thus continue the operation till the root of all the periods is found.

Example.—What is the cube root of 78314·6, true to two decimal places.

OPERATION.

	Col. I.	Col. II.	78314·600(42·78+.
1st term	4	16 × 4 =	64
2nd "	8	4800, 1st divisor)	14314
3rd "	122	5044 × 2 =	10088
4th "	124	529200, 2nd divisor)	4226600
5th "	1267	538069 × 7 =	3766483
6th "	1274	54698700, 3d divisor)	460117000
7th "	12818	54801244 × 8 =	438409952

Explanation.—The cube root of the greatest cube in 78 is 4, which i placed in the root and also in column I, then multiplying this 4 by itself give us 16 which is the 1st term in column II, and again multiplying this 16 by gives us 64, the number which we are to subtract from the first period 78.

Subtracting and bringing down the next period 314 we get 14314 for th next dividend.

Now adding 4, the figure placed in the root, to 4 the 1st term in col. I. give us 8, the 2nd term in col. I, multiplying this 8 by the 4, i. e., the figure in th root, gives us 32 which we add to the 1st term of col. II, and affix two ciphers We thus obtain 4800 the second term of col. II, which is our trial divisor.

We then find that 4800 goes 2 times in the dividend. This 2 we place i the root and also to the right of the sum of the 1st and 2nd terms of col. I. Th 1st and 2nd terms of col. I, added together make 12 and the 2 of the root affixe makes 122, the third term of col. I Then we multiply this 122 by 2, the la digit put in the root, this gives us 244 which we add to 4800, the second term col. II. and thus obtain 5044, the 3rd term. Lastly this third term multiplie by 2, gives us the number to subtract.

Note.—For examples in this method work any of the preceding question

APPLICATION OF THE CUBE ROOT.

33. *Principles Assumed.—I. Spheres are to one another as th cubes of their diameters.*

II. Cubes and all other regular solids are to one another as th cubes of their like dimensions.

Exercise 144.

1. If a cannon ball 3 inches in diameter weighs 8 lbs., what will b the weight of a ball of the same metal 4 inches in diameter?
   $3^3 : 4^3 :: 8$ lbs. : *Ans.* $= 18\frac{26}{27}$ lb
2. If a ball 3 inches in diameter weighs 4 lbs., what will be the weig of a ball that is 6 inches in diameter? *Ans.* 32 lb
3. If a globe of gold one inch in diameter be worth $120, what is th value of a globe 3½ inches in diameter? *Ans.* $514
4. If the weight of a well proportioned man, 5 feet 10 inches in heig be 180 pounds, what must have been the weight of Goliath Gath, who was 10 feet 4½ inches in height? *Ans.* 1015·1 lb

5. A person has a cube of clay whose sides are 973 ft. long; he wishes to take out of the same 5 cubes whose sides are 45 feet, 62 feet, 30 feet, 80 feet, and 20 feet. He requires to know the length of the side of the cube that can be formed out of the remaining clay. *Ans.* 972·69 ft.

6. What is the side of a cube which will contain as much as a chest 8 feet 3 inches long, 3 feet wide, and 2 feet 7 inches deep? *Ans.* 47·9843 inches.

7. Four ladies purchased a ball of exceeding fine thread, 3 in. in diameter. What portion of the diameter must each wind off so as to share off the thread equally?

Ans.	1st lady	must wind	off	·27432	inches.
	2nd	"	"	·34458	"
	3rd	"	"	·49122	"
	4th	"	"	1·88988	"

Note.—This question is solved by a method similar to that adopted in Example 18, Exercise 140.

EXTRACTION OF THE ROOTS OF HIGHER ORDERS.

34. When the index of the root is a power of 2 or 3, or a multiple of any power of 2 by any power of 3—

RULE.

Resolve the given index into its prime factors.

Extract the root denoted by one of these factors, then of this root, extract the root denoted by another factor, and so on till all the prime factors be used.

Thus, for the 4th root extract the square root of the square root.
for the 6th root extract the cube root of the square root.
for the 8th root extract the square root of the square root of the square root.
for the 12th root extract the cube root of the square root of the square root.
for the 16th root extract the square root four times.
for the 18th root extract the cube root of the cube root of the square root, &c., &c.

Exercise 145.

1. What is the fourth root of 19987173376? *Ans.* 376.
2. What is the sixth root of 308915776? *Ans.* 26.
3. Extract the ninth root of 40353607. *Ans.* 7.
4. Extract the eighteenth root of 387420489. *Ans.* 3.
5. Extract the twenty-seventh root of 134217728. *Ans.* 2.

LOGARITHMS.

35. The Logarithm of a number is the index of the power to which it is necessary to raise a given root or *base*, in order to produce the given number.

36. The Base of a system of logarithms is the *fixed number* to which all the logarithms of that system belong as indices.

Thus $10^3 = 1000$; here 3 is called the logarithm of 1000, to the base 10. So also $2^5 = 32$; here 5 is called the logarithm of 32, to the base 2, &c., &c.

37. A System of Logarithms is a collection of the logarithms of a series of numbers corresponding to the same base.

Any number whatever may be taken as the base of the system; but it is obvious that some numbers are much more convenient than others.

38. Two systems of logarithms have been constructe and tables calculated with great care. They are—

1st. The Common System or Briggean System, whos base is 10.

2nd. Napierian System, whose base is 2·71828.

The Napierian System was invented by Baron Napier, and the peculiar ba 2·71828, was adopted chiefly because the logarithms having that base are mor simply expressed and more easily calculated than any other. It has hence bee called the *Natural System* of Logarithms. These Logarithms were also fo merly called *Hyperbolic* logarithms, from certain relations found to exist be tween them and the asymptotic spaces of the hyperbola, and which wer erroneously believed to be peculiar to them.

The Common System was shortly afterwards invented by Briggs a adopted by Baron Napier, and is the system now universally employed for t purposes of calculation.

39. The *Characteristic* of a logarithm is the part whic stands to the left of the decimal point.

40. The *Mantissa* (*handful*) is that part of the loga rithm which stands to the right of the decimal point.

41. Since 10 is the base of the common system o logarithms and at the same time the radix of our syste of notation, we have—

$100000 = 10^5$; whence log. $100000 = 5$
$10000 = 10^4$; whence log. $10000 = 4$
$1000 = 10^3$; whence log. $1000 = 3$
$100 = 10^2$; whence log. $100 = 2$
$10 = 10^1$; whence log. $10 = 1$
$1 = 10^0$; whence log. $1 = 0$
$·1 = 10^{-1}$; whence log. $·1 = -1$
$·01 = 10^{-2}$; whence log. $·01 = -2$
$·001 = 10^{-3}$; whence log. $·001 = -3$
$·0001 = 10^{-4}$; whence log. $·0001 = -4$

42. From this it appears that the logarithm of any number between 1 and 0 will be more than 0 and less than 1; i. e., will be a fraction or a decimal; so lso the logarithm of any number between 10 and 100 will be greater than 1 nd less than 2; i. e., will be 1 and a fraction, or a decimal; so also the logarithm of any number between 100 and 1000 will be 2 and a decimal, &c.

Hence, the characteristic of any number containing igits to the left of the decimal point is positive and numerically *one less* than the number of such digits.

Thus the characteristic of 7842 is 3; of 978·26 it is 2; of 9813426789 it is 9; 8·00429 it is 0; of 26789·426789 it is 4, &c.

43. It also appears, from Art. 41, that the logarithm of every number between 1 and ·1 will be less than 0 and greater than −1; that is, it will be equal −1, *plus* some decimal; the logarithm of every number between ·1 and ·01 ill be less than −1 and greater than −2; or, in other words, will be −2 *plus* me decimal; so also the logarithm of every number between ·01 and ·001 will −3 *plus* some decimal, &c., &c.

Hence, the characteristic of the logarithm of a decimal negative and numerically *one greater* than the number f 0s which come between the decimal point and the first gnificant figure.

Thus, the characteristic of the logarithm of ·000001 is $\bar{6}$; the characteristic the logarithm of ·00000000002347 is $\overline{11}$; the characteristic of the logarithm of 00278926345 is $\bar{4}$, &c., &c.

Note.—*The negative sign affects only the characteristic—the antissa or decimal portion of a logarithm is always positive. To dicate this it is customary to write the negative sign over the characistic, as in the above examples, and not before it.*

Exercise 146.

What are the characteristics of the logarithms of the following mbers:

723, 9126·4, 81234·567, 912678·96124567, 23·912342.

Ans. 2, 3, 4, 5, and 1.

·027, ·002134, ·000000698, ·8126714, ·0000000002134.

Ans. $\bar{2}$, $\bar{3}$, $\bar{7}$, $\bar{1}$, and $\overline{10}$.

1·1111111, 111111·11, 1000000000, ·000000002162, 7, 12·78.

Ans. 0, 5, 9, $\bar{9}$, 0, and 1.

44. Since (Art. 11), to divide one power of a number by another power of same we subtract the index of the divisor from the index of the dividend, since common logarithms are indices to the base 10, let us take the number 80 and successively dividing it by 10, examine the results.

Numbers.		Logarithms.
47280	....	= 4·674677
4728	....	= 3·674677
472·8	....	= 2·674677
47·28	....	= 1·674677
4·728	....	= 0·674677
·4728	....	= $\bar{1}$·674677
·04728	....	= $\bar{2}$·674677
·004728	....	= $\bar{3}$·674677

Here we have simply performed the same operation by two different methods, 1st, dividing the *numbers* by 10, and 2nd, from the *logarithms* corresponding to the numbers, subtracting 1, the logarithm of 10.

From this illustration it is evident that,—

1st. The characteristic of the logarithm of a number is dependent wholly upon the position of the decimal point in that number, and is not at all affected by the sequence of the digits that compose that number; and

2nd. The Mantissa or decimal part of the logarithm of a number is dependent wholly upon the sequence of the digits that compose that number, and it is not at all affected by the position of the decimal point.

Note.—It is only common logarithms (i. e., those having 10 for their base) that possess the important property of having the same mantissa for the same figure, whether integral or decimal, or both, and it was this property that induced Briggs to adopt that base in preference to the Napierian base, 2·71828.

45. Since the characteristic of the logarithm of any number does not depend upon the value of the digits composing that number, and is so easily found by attention to the rules found in Arts. 42, 43. it is customary to omit it altogether in logarithmic tables, and merely give the mantissa.

The annexed tables contain the logarithms of all numbers from 1 to 10000 calculated to 6 decimal places. When greater accuracy is required, tables calculated to a greater number of places are used. By means of the proportional parts and difference given in the tables, the logarithm corresponding to all numbers whatever, may be found with sufficient accuracy for all practical purposes.

46. To find the logarithm of any number not greater than 100—

RULE.

Find on the first page of the table of logarithms, the given number in the column marked No., and directly opposite to it,—in the column marked log., will be found the logarithm.

Example 1.—What is the logarithm of 47? *Ans.* 1·672098.

Note.—By saying that 1·672098 is the logarithm of 47, we simply mean that the base 10, raised to the power 1·672098, is equal to 47, or briefly $10^{1\cdot672098} = 47$.

Example 2.—What is the logarithm of 93? *Ans.* 1·968483.

---

47. To find the logarithm of any number consisting of not more than four digits—

RULE.

Find, in the column marked N, the first three digits of the given nnmber.

Then the mantissa will be found in the intersection of the horizontal line containing these three digits and the vertical column at the head of which stands the fourth digit.

To this mantissa attach the characteristic as found by the rules in Arts. 42, 43.

EXAMPLE 1.—What is the logarithm of 7983?

Looking in the column marked N, we find the first three digits 798, on page 398 in the fourth horizontal division, counting from the top of the page and in the last line but one of that division. Carrying the eye along this horizontal line till we come to the vertical column, at the head of which stands the remaining digit, 3, we obtain for the mantissa of the required logarithm ·902166, to which we prefix the characteristic 3 (since there are four digits to the left of the decimal point in the given number), and thus obtain the required logarithm 3·902166.

EXAMPLE 2.—What is the logarithm of ·0000001234?

The first three digits, viz: 123, are found in the fourth line of the third horizontal division on page 332, and at the intersection of this line with the column headed 4, is found ·091315. To this we attach the characteristic $\bar{7}$, (since there are *six* 0s, between the decimal point and the first significant figure) and thus obtain the required logarithm, $\bar{7}$·091315.

EXERCISE 147.

1. What are the logarithms of 5794, 57·94, 5794000, and ·0005794?

Ans. 3·762978, 1·762978, 6·762978, and $\bar{4}$·762978.

2. What are the logarithms of 1·169, 11690, and $\frac{1169}{10000000}$?

Ans. 0·067815, 4·067815, and $\bar{3}$·067815.

3. What are the logarithms of ·734, 7340000000, and ·00000000734?

Ans. $\bar{1}$·865694, 9·865696, and $\bar{9}$·865696.

4. What are the logarithms of 978·4, 9·784, 978400, and ·9784?

Ans. 2·990516, 0·990516, 5·990516, and $\bar{1}$·990516.

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48. To find the logarithm of a number containing more than four digits—

RULE.

FIRST METHOD.—*Find the mantissa corresponding to the logarithm of the first four digits by the last rule. Subtract this mantissa from the next following mantissa in the tables. Multiply the difference thus obtained by the remaining digits of the given number, and cut off from the product as many digits as there were in the multiplier (but at the same time adding unity if the highest cut off be not less than 5).*

Add the number thus obtained to the mantissa of the logarithm corresponding to the first four digits, and the result will be the mantissa of the given number.

Lastly attach the characteristic to this mantissa.

EXAMPLE 1.—What is the logarithm of 53803·2?

OPERATION.

The mantissa of the logarithm of 5380 (the first four digits) is ·730782 and the next following mantissa is ·730863.

Then from ·730863
Subtract ·730782

Difference 81; and 81 × 32 (remaining digits of given number)

=2502, from which we cut off *two* digits, since we multiplied by a number of two digits, and since the highest digit cut off is not less than 5, we add unity to the part retained, which gives us 26.

Then mantissa of logarithm of first four digits ·730782
Add 26

Mantissa of logarithm of given number ·730808
To which attach the characteristic 4 and required logarithm=4·730808.

NOTE.—Except at the beginning of the tables, where the mantissas increase rapidly in magnitude, the difference may be taken from the right hand column, (headed D) and opposite the first three digits of the given number, where the mean difference of the mantissas in that line will be found.

EXAMPLE 2.—What is the logarithm of 882·17242?

OPERATION.

Mantissa of logarithm of 8821 .... ·920176
Difference from column D=52; and 52×7242=376584 from which we cut off four digits and add 38

·920214

To which we attach the characteristic 2 and required logarithm=2·920214

49. The difference given in the column headed D in the tables, is that due to an increment of *one unit* in the fourth figure of natural number, thus

Logarithm of 5738 .... 3·758761
Logarithm of 5739 .... 3·758836

Difference of natural numbers=1; difference of logarithms=75

And since it is shown in common works on Algebra that, with small increments in the natural numbers the logarithms corresponding to them increase in arithmetical progression, in order to find the logarithm of any number between those given above, we consider that the increment of the logarithm to be added to 3·758761, bears the same proportion to 75 (the increment for 1), that the increment of the natural number does to 1.

For example.—Let it be required to find the logarithm of 5738·47.

Here the increment of the given number being ·47, we form the proportion 1 : ·47 : : 75 : ·47×75=35·25, the increment to be added to 3·758761, and this addition having been made, we get 3·758796 for the logarithm of 5738·47.

Similarly, if the increment of the natural number had been ·047 or ·0047, the corresponding increment of the log. would have been 3·525 or ·3525.

These illustrations sufficiently explain the reasons of the last rule.

50. Taking the same number as in the last article and dividing the difference 75 by 10, we obtain 7·5 the difference corresponding to an increase of one unit in the *fifth* place of the natural number; the double of this, or 15 for two units, the treble or 22·5 for the three units, and so on; and each of the numbers thus obtained will be the increment of the logarithm corresponding to an increase of that number of units in the *fifth* place of the natural number. The increments thus obtained, and corresponding to each of the nine digits, are inserted in the left hand column of the tables, headed P. P. (Proportional Parts).

51. The numbers in the column headed P. P., as already explained, are the increments in the logarithm for an increase in the *fifth* place of the natur numbers. They express also the increments for the digits in the *sixth*, *seventh*, *eighth*, *ninth*, &c., places of the natural number, when they are divided by 1 100, 1000, &c., as the case may be.

52. Hence to find the logarithm of any number containing more than four digits—

RULE.

Second Method.—*Find the mantissa of the logarithm corresponding to the first four digits of the given number.*

Find in the same horizontal division as that in which the mantissa is found, the proportional part in the column headed P. P., corresponding to the digit in the fifth place of the given number, and set it down beneath the part of the mantissa already found, so that their right hand digits may be in the same vertical line. Find the P. P. corresponding to the digit in the sixth place of the given number, and set it down so that its right hand figure may be one place to the right of the last. Find the P. P. corresponding to the digit in the seventh place of the given number and set it down one place to the right of the last, and so on till all the digits of the given number be used.

Add the part of the mantissa already found, and the P. Ps. as written, together, and reject from the result all but the first six digits to the left, adding one to the last retained, if the highest of the rejected digits be not less than 5—the result will be the mantissa of the logarithm of given number.

Lastly, attach the proper characteristic to this mantissa, and the result will be the required logarithm.

Example 1.—What is the logarithm of 8372·468 ?

OPERATION.

Mantissa of logarithm of 8372 = ·922829
P. P. corresponding to ·4 = 21
P. P. " to ·06 = 31
P. P. " to ·008 = 42

Sum = ·922853|52

Therefore required mantissa = ·922854 and required log. = 3·922854.

Example 2.—What is the logarithm of 408567 ?

OPERATION.

Mantissa of logarithm of 408500 = ·605844
P. P. corresponding to 60 = 64
P. P. " to 7 = 75

Sum = ·6059155

Therefore required logarithm is 5·605916.

Exercise 148.

FIND THE LOGARITHMS OF THE FOLLOWING NUMBERS BY THE FIRST METHOD—OBTAINING THE DIFFERENCES BY SUBTRACTION.

1. What are the logarithms corresponding to 8193217, 73·9245, and ·843742? *Ans.* 6·913455, 1·868789, and $\bar{1}$·926210.

2. Find the logarithms corresponding to ·000234564 and ·001007013. *Ans.* $\bar{4}$·370261 and $\bar{3}$·003035.

USING THE TABULAR DIFFERENCES.

3. Find the logarithms corresponding to 52·376 and 129·476.
Ans. 1·719133 and 2·112189.

USING THE PROPORTIONAL PARTS.

4. Find the logarithms corresponding to ·000471398 and 9136712.
Ans. $\bar{4}$·673387 and 6·960790.

5. Find the logarithms corresponding to 4·23429 and 763·12987.
Ans. 0·626780 and 2·882598.

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53. To find the logarithm of a vulgar fraction—

RULE.

Subtract the logarithm of the denominator from the logarithm of the numerator.

54. To find the logarithm of a mixed number—

RULE.

Either reduce the mixed number to a fraction and proceed as in Art. 53, *or reduce the fractional part to a decimal, attach it to the whole number and proceed as in Arts.* 48–52.

55. To find the natural number corresponding to any given logarithm.—

RULE.

FIRST METHOD.—*Find the logarithm in the table which is next lower than the given one, and the four digits corresponding to it will be the first four digits of the required number.*

II. Subtract this logarithm from the given logarithm, to the remainder annex one cipher and divide by the tabular difference corresponding to the four digits already obtained, the quotient will be the fifth digit.

III. To the remainder attach another cipher and again divide by the tabular difference, the quotient will be the sixth digit, and thus proceed till a sufficient number of digits has been obtained.

IV. The characteristic of the logarithm shows where to place the decimal point.

NOTE.—The number cannot be carried with accuracy to more places than the logarithm has decimal places. (See Art. 56.)

EXAMPLE 1.—Find the number corresponding to the logarithm 4·923267.

OPERATION.

Given log. ·923267
Next lower in tables, ·923244 = log. of 8380.
Difference = 23 Tabular difference = 52.
Then 23000 ÷ 52 gives 442 for digits in 5th, 6th, and 7th places.

Hence the digits of the natural number are 8880442; and since the characteristic is 4, *i. e. one less* than the number of digits to the left of the decimal point, the required number is 88804·42.

Second Method.—*Find the first four digits of the required number and also the difference between the given logarithm and the next lower in the table as in the last rule.*

II. *Find in the same horizontal division of the table the highest P. P. that does not exceed this difference. Opposite to it in the column headed N. will be found the digit of the fifth place.*

III. *Subtract this P. P. from the difference, to the remainder annex one cipher and find the highest P. P. not exceeding the number thus formed. Opposite to it in column N. will be found the sixth digit.*

IV. *Continue this process by the addition of ciphers, till the required number of digits be found.*

Example 2.—Find the natural number corresponding to the logarithm 3·553259.

OPERATION.

Given log. ·553259
Next lower in the table ·553155 = log. of 3574
Difference = 104
Highest P. P. not greater than 104 = 98 corresponds to 8 for *fifth* place.
60
Highest P. P. not greater than 60 = 49 corresponds to 4 in *sixth* place.
Highest P. P. not greater than 110 = 110 corresponds to 9 in *seventh* [place.
110

Therefore digits of required number are 3574849; and since the characteristic is 3, there must be four digits to the left of the decimal point.
Hence required number is 3574·849.

Exercise 149.

BY FIRST METHOD.

1. Find the natural numbers corresponding to the logarithms 4·137139, 0·718134 and $\bar{4}$·635421.
Ans. 13713·227, 5·225578 and ·0004319376.

2. Of what numbers are 2·921686 and $\bar{1}$ 922165 the logarithms?
Ans. 835 and ·8359211.

BY SECOND METHOD.

. Of what numbers are 5·407968, 7·408386 and $\bar{3}$·416369 the logarithms?
Ans. 255839·4, 25608588 and ·0026088.

. What are the natural numbers corresponding to the logarithms 4·877777 and 0·555555?
Ans. 75470·5168 and 3·5938.

56. In order to ascertain how many figures of these results may be relied upon as correct, let us take from the tables any logarithm, as 4·285685.

Now the real value of this logarithm if carried to a greater number of places might be anything between 4·2856835 and 4·2856845, and might therefore differ from the given logarithm by very nearly ·0000005, which is therefore the extreme limit of the error attached to tables of six places; i. e. any difference less than ·0000005 might occur without producing any change in the logarithm as given in the table.

Now it is demonstrated in works treating of the theory of logarithms that the difference between the logarithms of numbers, which differ only by unity, is less than the modulus of the system divided by the smaller number. The modulus of the common system of logarithms is ·4342945, and if we let n represent the smaller number, the difference between the logarithms of n and of $n+1$ is less than $\cdot 4342945 \div n$.

Now we have shown that the difference between the true logarithm and that given in the table to six places, may be nearly equal to ·0000005, which is therefore less than $\cdot 4342945 \div n$, or n is less than $\frac{\cdot 4342945}{\cdot 0000005}$ But $\frac{\cdot 4342945}{\cdot 0000005} = 868589$. That is, unless the number whose logarithm is given be less than 868589 its value cannot be found accurately beyond the first *five* digits, but if it be less than 868589, then the first *six* figures found from the table will be correct.

If tables of seven or eight places are used, the result can be depended on to seven or eight places, if the number be less than 868589 or if the mantissa be less than ·9378; but if greater, then the result can be relied on only to one less number of figures than the decimals of the logarithm.

LOGARITHMIC ARITHMETIC.

57. The *Arithmetical Complement* of a logarithm is the remainder obtained by subtracting the logarithm from 10.

Thus the arithmetical complement of 2·718426 is 10—2·718426 = 7·286574.

EXERCISE 150.

1. Find the arithmetical complements of 5·631642 and 0·714000.
Ans. 4·368358 and 9·286000.

2. Find the arithmetical complements of $\bar{3}$·123456 and $\bar{7}$·213149.
Ans. 12·876544 and 16·786851.

3. Find the arithmetical complements of 6·124357 and $\bar{2}$·000837.
Ans. 3·875643 and 11·999163.

58. To multiply two or more numbers together by means of logarithms :—

RULE.

I. *Add their logarithms and the sum will be the logarithm of their product.*

II. *Find the natural number corresponding to this logarithm.*

NOTE 1.—For reason see Art. 10.

NOTE 2.—The following exercises are all worked by the difference, and not by the proportional parts:

EXAMPLE.—Multiply 5681 by 47.

Logarithm of 5681=3·750586
" " 47=1·672098

5·422684
5·422590=logarithm of 264600

94= 57

Ans. 264657

EXERCISE 151.

1. Multiply 61, 22, and 65 together. *Ans.* 87230.
2. Multiply 52, 734, and 6 together. *Ans.* 229008.
3. Multiply together 35·36, 2·1046, ·8372 and ·00294. *Ans.* ·185761.
4. Multiply ·00008764 by ·86359. *Ans.* ·000075685.

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59. To divide numbers by means of their logarithms—

RULE.

I. Subtract the logarithm of the divisor from the logarithm of the dividend: the result will be the logarithm of the required quotient.

II. Find the natural number corresponding to this.

NOTE.—for reason see Art. 11.

EXAMPLE 1.—Divide 6732·7 by 478.

OPERATION.

Logarithm of 6732·7=3·828189
Logarithm of 478 =2·679428

Difference=1·148761
1·148608=logarithm of 14·0800

153= 51

Ans. 14·0851

EXAMPLE 2.—Divide ·036584 by ·00078598.

OPERATION.

Logarithm of ·036584=$\bar{2}$·563291
Logarithm of ·00078598=$\bar{4}$·895384

Difference=1·667907
1·667826=logarithm of 46·5400
81= 87

Ans. 46·5487

60. *Instead of subtracting the logarithm of the divisor, we may add its arithmetical complement—the result, with 10 subtracted from the characteristic, will be the logarithm of the quotient.*

Thus, in the last example the arithmetical complement of $\bar{4}$·895384 is 13·104616, and this added to $\bar{2}$·563291 gives 11·667907, and subtracting 10 from this characteristic, gives us 1·667907, the same as obtained by the other method.

NOTE.—This method of using the arithmetical complement is very convenient when we have to divide one number by the product of several others.

EXERCISE 152.

1. Divide ·6734 by ·0009278. *Ans.* 725·8033.
2. Divide 437·89 by 62·735. *Ans.* 6·98.
3. Divide 93·217 by ·0007132. *Ans.* 130702·4.
4. Divide 9835267 by the product of 23, 189 and 2·748. *Ans.* 823·339.

---

61. To raise a quantity to any power by means of logarithms—

RULE.

I. Multiply the logarithm of the given number by the index of the required power, the result will be the logarithm of the required power.

II. Find the natural number corresponding to this logarithm.

NOTE.—For reason see Art. 12.

EXAMPLE 1.—Find the 10th power of 2.

OPERATION.

Logarithm of 2=0·301030.

0·301030 × 10 = 3·010300 = logarithm of 1024. *Ans.*

EXAMPLE 2.—Find the 7th power of 2·71.

OPERATION.

Logarithm of 2·71 = 0·432969.

Then 0·432969 × 7 = 3·030783 = logarithm of 1073·45. *Ans.*

NOTE.—In order to obtain the correct result when the characteristic happens to be negative, it must be recollected that the mantissa is *always* positive.

EXERCISE 153.

1. What is the 5th power of 5? *Ans.* 3125.
2. What is the 6th power of 1·073? *Ans.* 1·5261.
3. What is the 4th power of ·0279? *Ans.* ·00000060592.
4. What is the 11th power of 1·111? *Ans.* 3·1831.

---

62. To extract any root of a given number by means of logarithms—

RULE.

I. Find the logarithm of the given number and divide it by the index of the required root, the result will be the logarithm of the root.

II. Find the natural number corresponding to this logarithm.

Note.—For reason see Art. 15.

Example.—What is the cube root of 12345?

OPERATION.

Logarithm of 12345=4·091491.
Then 4·091491÷3=1·363830=logarithm of 23·11159. *Ans.*

63. To extract any root when the characteristic of the logarithm of the given number is negative:—

RULE.

I. If the characteristic is exactly divisible by the divisor, divide in the ordinary way, but make the characteristic of the quotient negative.

II. If the negative characteristic is not exactly divisible, add what will make it so, both to it and to the decimal part of the logarithm. Then proceed with the division.

Example 22.—Extract the fourth root of ·0076542.

OPERATION.

Logarithm of ·0076542=$\overline{3}$·883899.

Now since $\overline{3}$ is not exactly divisible by 4 we add−1 to the characteristic and + 1 to the mantissa which gives us $\overline{4}$ + 1·883899 and this is evidently = $\overline{3}$·883899.

Then $\overline{4}$+1·883899÷4=$\overline{1}$·4709747=logarithm of ·295784. *Ans.*

EXERCISE 154.

1. Extract the 7th root of 913426000. *Ans.* 19·0588.
2. Extract the 11th root of 1·61342. *Ans.* 1·04444.
3. Extract the 5th root of ·000007139. *Ans.* ·0934817.
4. Extract the 7th root of ·002147. *Ans.* ·41575.

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64. When the logarithms of two or more prime numbers are given, the logarithm of any multiples of these factors by each other can be easily obtained by attention to the foregoing rules.

Thus if the logarithm of 2 and 3 be given:—

1st. We can obtain the logarithm of any power of 2 or 3 by Art. 61, and any root of 2 or 3 by Art. 62.

2nd. We know the logarithm of 10 to be 1, and hence we can obtain the logarithm of 5, since 10÷2=5 and also of 3·$\dot{3}$ since 10÷3=3·$\dot{3}$, hence we can also obtain the logarithm of any power or root of 5 or 3·$\dot{3}$.

3rd. By Arts. 58, 59, we can obtain the logarithm of any power or root of 2, 3, 5 and 3·$\dot{3}$ multiplied by any power or root of 2, 3, 5 or 3·$\dot{3}$.

Example.—Given the logarithm of 2 = 0·301030 and the logarithm of 3 = 0·477121. Find the logarithms of 500, 24, 54, 120, 75000, 16⅔, ⅓, and 13·5.

OPERATION.

Since 5=10÷2 the logarithm of 5=log. 10−log. 2=1−0·301030=0·698970.
Then logarithm of 500=2·698970.

$24=8\times3=2^3\times3$ ∴ log. 24=(log. 2)×3+(log. 3.)
log. 2=0·301030×3=0·903090
log. 3= ·477121
Sum=1·380211=log. 24.

$54=27\times2=3^3\times2$ ∴ log. 54=(log. 3)×3+(log. 2.)
log. 3=0·477121×3=1·431363
log. 2= 0·301030
Sum=1·732393=log. 54.

$120=4\times3\times10=2^2\times3\times10$ ∴ log. 120=(log. 2)×2+(log. 3)+(log. 10.)
log. 2=0·301030×2=0·602060
log. 3= 0·477121
log. 10= 1
Sum=2·079181=log. 120.

$75000=25\times3\times1000=5^2\times3\times1000$ ∴ log. 75000=(log. 5)×2+(log. 3) + (log. 1000.)
log. 5=0·698970×2=1·397940
log. 3= 0·477121
log. 1000= 3
Sum=4·875061=log. 75000.

$16\frac{2}{3}=3{\cdot}\dot{3}\times5$ ∴ logarithm of $16\frac{2}{3}$=(log. $3{\cdot}\dot{3}$)+(log. 5.)
Since 10÷3=$3{\cdot}\dot{3}$, log. $3{\cdot}\dot{3}$=log. 10−log. 3=1−0·477121=0·522879
logarithm 5= 0·698970
Sum=1·221849=log. $16\frac{2}{3}$.

$\frac{1}{2}={\cdot}5$ ∴ by changing only the characteristic $=\bar{1}{\cdot}698970$ = logarithm $\frac{1}{2}$.
$13{\cdot}5={\cdot}5\times27={\cdot}5\times3^3$ ∴ logarithm 13·5=(log. 3)×3+(log. ·5)
logarithm 3=0·477121×3=1·431363
logarithm ·5= $\bar{1}{\cdot}698970$
Sum=1·130333=log. 13·5.

Exercise 155.

1. Given logarithm 2 = 0·301030 and log. 7 = 0·845098, find the logarithms of 14000, 4·9, ·00196, 1750, $1428{\cdot}\dot{5}7142\dot{8}$, ·00000112 and 3·0625.

Ans. Log. 14000 = 4·146128.
Log. 4·9 = 0·690196.
Log. ·00196 = $\bar{3}{\cdot}292256$.
Log. 1750 = 3·243038.
Log. $1428{\cdot}\dot{5}7142\dot{8}$ = 3·154902.
Log. ·00000112 = $\bar{6}{\cdot}049218$.
Log. 3·0625 = 0·486076.

Note.—$1428{\cdot}\dot{5}7142\dot{8}=\frac{1}{7}\times10000$, also 3·0625=49÷16.

EXAMPLE 2.—Given logarithm $\frac{1}{2}=\bar{1}{\cdot}698970$
logarithm $3=0{\cdot}477121$
logarithm $11=1{\cdot}041393$

Find the logarithms of 49½, 363, $4{\cdot}\dot{0}\dot{9}$, $2{\cdot}\dot{4}$, $392{\cdot}\dot{7}\dot{2}$, 293333⅓, and 19·965.

Ans. Logarithm of 49½ = 1·694605.
Logarithm of 363 = 2·559907.
Logarithm of $4{\cdot}\dot{0}\dot{9}=0{\cdot}611819$.
Logarithm of $2{\cdot}\dot{4}=0{\cdot}388181$.
Logarithm of $392{\cdot}\dot{7}\dot{2}=2{\cdot}594090$.
Logarithm of 293333⅓ = 5·467362.
Logarithm of 19·965 = 1·300270.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the numbered articles of the section.*

1. What is the power of a number? (1)
2. What is the root of a number? (2)
3. Why is the second power of a number called its square? (4)
4. Why is the third power of a number called its cube? (5)
5. What is the index or exponent of a power? (6)
6. What is involution? (8)
7. How do we multiply two or more different powers of the same number together? (10)
8. How do we divide any power of a number by another power of the same number? (11)
9. How do we find any required power of a given power? (12)
10. What is evolution? (13)
11. By what methods do we indicate a root of a number? (14)
12. How do we extract any root of a given power of a number? (15)
13. What is meant by extracting the square root of a number? (16)
14. What is the first step in extracting the square root of a number? (16)
15. Why do we point off into periods of two figures each? (18-I)
16. What is the second step in the process of extracting the square root? (16)
17. How do we know that the square root of the highest square in the left hand period is the highest digit of the root? (18-II)
18. What is the third step in the process of extracting the square root? (16)
19. Why do we bring down only the next period to the right? (18-II in Ex. 2)
20. What is the fourth part of the process for extracting the square root? (16)
21. Why do we double the part of the root already found for a trial divisor? (18-III)
22. What is the next step in extracting the square root of a number? (16)
23. Why do we not include the right hand figure of the dividend when seeking how many times the trial divisor is contained in it? (18-IV)
24. Why do we place the digit thus found in both the divisor and the root? (18-V)
25. What are the other steps used in extracting the square root? (16)
26. How do we extract the square root of a decimal? (19)

27. How do we extract the square root of a fraction or mixed number? (20)
28. What is a triangle? (22) What is a right-angled triangle? (23)
29. How may any one side of a right-angled triangle be found when the other two are given? (24)
30. What proportion exists between different circles? (25)
31. How may the area of a circle be found when the diameter is known? (25)
32. What is meant by extracting the cube root of a number? (26)
33. Give the different steps of the process of extracting the cube root. (26)
34. If a number consist of a certain number of tens, plus a certain number of units, of what does its cube consist? (27)
35. Why do we divide off into periods of three figures each? (28-I)
36. How do we know that the cube root of the highest cube contained in the left hand period is the highest digit of the root? (28-II)
37. Whence do we obtain, in the cube root, the constant multipliers 300 and 30. Illustrate by an example. (28 IV, and VI)
38. Why do we make the two additions, indicated in the rule, to the trial divisor? (28-VI)
39. How do we extract the cube root of a decimal? (29)
40. How do we extract the cube root of a fraction or mixed number? (30)
41. In extracting the cube root of a number in any other scale, what changes must we make in the rule? (31)
42. Give the different steps of Horner's method of extracting the cube root (32)
43. What proportion exists between the magnitude of similar solids? (33)
44. How do we extract the higher roots when the index is a power of 2 or 3 or a multiple of 2 by 3? (34)
45. What is a logarithm? (35)
46. What is the base of a system of logarithms? (36)
47. What is a system of logarithms? (37)
48. What systems of logarithms have been constructed and how do they differ from one another? (38)
49. What is the characteristic of a logarithm? (39)
50. What is the decimal part of the logarithm called? (40)
51. How do we find the characteristic of a logarithm? (42 and 43)
52. Why is the negative sign written *over* the characteristic of the logarithm of a decimal? (43, Note)
53. Show that the characteristic of the logarithm of a number depends only on the position of the decimal point in the number, and the mantissa only in the sequence of figures. (44)
54. Explain clearly what is meant by the numbers in column D of the tables. (49)
55. Explain how the proportional parts in column P. P. are obtained. (50)
56. Explain how the numbers in the column headed P. P. become the increments to be added to the logarithms for an increase in the sixth, seventh, eighth, &c., place in the natural number. (51)
57. How do we find the logarithm of a vulgar fraction? (53)
58. Explain to how many figures we may rely upon the accuracy of the results obtained by logarithmic tables. (56)
59. What is the arithmetical complement of a logarithm? (57)
60. How do we multiply numbers by means of their logarithms? (58)
61. How do we divide numbers by means of their logarithms? (59, 60)
62. How do we involve and evolve quantities by means of logarithms? (61, 62, 63)

SECTION XI.

PROGRESSION, POSITION, COMPOUND INTEREST, AND ANNUITIES.

PROGRESSION.

1. Quantities are said to be in Arithmetical Progression when they increase or decrease by a *common difference.*

Thus, 2, 5, 8, 11, 14, &c., are in arithmetical progression, the common difference being 3.

12, 10, 8, 6, &c., are in arithmetical progression, the common difference being 2.

2. In every progression the first and the last terms are called the *extremes*, and the intermediate terms the *means.*

ARITHMETICAL PROGRESSION.

3. In *arithmetical progression* there are five things to be considered ;

1. *The first term.*
2. *The last term.*
3. *The common difference.*
4. *The number of terms.*
5. *The sum of the series.*

These quantities are so related to one another that any three of them being given, the other two can be found, and hence there are 20 distinct cases arising from these combinations.

4. If we represent these five quantities by letters, thus:

a = *the first term.*
l = *the last term.*
d = *the common difference.*
n = *the number of terms.*
s = *the sum of the series.*

We shall be able easily to deduce algebraic formulæ which, being interpreted, become the common arithmetical rules for arithmetical progression.

5. The general expression for an arithmetical series then becomes

$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + (a+5d) + \text{, \&c.}$$

where the coefficient of d is always 1 *less than the number of the terms.* Thus in the third term the coefficient of d is 2, which is 1 less than the number of the term: in the *fifth* term the coefficient of d is 4, which is 1 less than the number of the term, &c.

Hence $l = a + (n-1)d$; that is, the *last term* of an arithmetical series is equal to the *first term* added to the product of the *common difference* by *one less than the number of terms.*

6. Since the sum of the series is equal to the sum of all the terms taken in any order whatever, we have

$$\begin{array}{rrr|l|l|l|l|l|l|l}
 & s = & a+ & a+d+ & a+2d+ & a+3d+ & ..\ l-3d+ & l-2d+ & l-d+ & l \\
\text{Also} & s = & l+ & l-d+ & l-2d+ & l-3d+ & ..\ a+3d+ & a+2d+ & a+d+ & a
\end{array}$$

Hence $2s = (a+l)+(a+l)+(a+l)+(a+l)+\ldots\ldots$ to n terms.
But $(a+l)+(a+l)\ldots\ldots$ to n terms $= (a+l)n$.

Therefore $2s=(a+l)n$, and dividing these equals by 2, we have $s=(a+l)\frac{n}{2}$

That is, *the sum* of the series is found by adding together the *first* and *last terms* and multiplying *their sum* by *half the number of terms.*

NOTE.—In adding the corresponding terms of the foregoing series together the d's cancel out, thus adding the second terms of the right hand members together we have $a+d+l-d$, where the d's cancel, and the sum becomes $a+l$: so also in the third terms we have $a+2d+l-2d = a+l$, &c.

7. From the formula obtained in Art. 5, we find by transposing the terms

$$l = a+(n-1)d$$

$$a = l-(n-1)d$$

$$d = \frac{l-a}{n-1}$$

$$n = \frac{l-a}{d} + 1$$

and substituting these values of l, a, d, and n in the formula obtained in Art. 6, we find

$$s = \left\{2a+(n-1)d\right\}\frac{n}{2}$$

$$s = \left\{2l-(n-1)d\right\}\frac{n}{2}$$

$$s = \left\{\frac{(l-a)\,(l+a)}{2d} + \frac{l+a}{2}\right.$$

We thus obtain the five fundamental formulas from which the others are derived by transposing the terms, &c. Thus,

$$\begin{array}{lll}
l = a+(n-1)d & \text{gives formulas for} & l,\ a,\ n,\ d = 4 \\
s = (a+l)\frac{n}{2} & \text{"} & s,\ a,\ l,\ n = 4 \\
s = \left\{2a+(n-1)d\right\}\frac{n}{2} & \text{"} & s,\ a,\ n,\ d = 4 \\
s = \left\{2l-(n-1)d\right\}\frac{n}{2} & \text{"} & s,\ l,\ n,\ d = 4 \\
s = \frac{(l+a)\,(l-a)}{2d} + \frac{l+a}{2} & \text{"} & s,\ a,\ l,\ d = 4 \\
 & & \text{Total,}\quad 20
\end{array}$$

8. THE FOLLOWING TABLE GIVES THE 20 FORMULAS FOR ARITHMETICAL PROGRESSION WITH THEIR RELATIONS, &c.

No.	Given.	Required.	Formulas.	Whence derived.
I.	a, d, n	l	$l = a+(n-1)d$	fundamental.
II.	a, d, s		$l = -\frac{1}{2}d + \sqrt{2ds+(a-\frac{1}{2}d)^2}$	VIII.
III.	a, n, s		$l = \frac{2s}{n} - a$	V.
IV.	d, n, s		$l = \frac{s}{n} + \frac{(n-1)d}{2}$	VII.
V.	a, l, n	s	$s = (a+l)\frac{n}{2}$	fundamental.
VI.	a, d, n		$s = \left\{2a+(n-1)d\right\}\frac{n}{2}$	V. and I.
VII.	d, l, n		$s = \left\{2l-(n-1)d\right\}\frac{n}{2}$	V. and XVII.
VIII.	a, d, l		$s = \frac{(l+a)(l-a)}{2d} + \frac{l+a}{2}$	V. and XIII.
IX.	a, n, l	d	$d = \frac{l-a}{n-1}$	I.
X.	a, n, s		$d = \frac{2s-2an}{n(n-1)}$	VI.
XI.	a, l, s		$d = \frac{(l+a)(l-a)}{2s-l-a}$	VIII.
XII.	l, n, s		$d = \frac{2nl-2s}{n(n-1)}$	VII.
XIII.	a, d, l	n	$n = \frac{l-a}{d} + 1$	I.
XIV.	a, d, s		$n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d}+\left(\frac{2a-d}{2d}\right)^2}$	VI.
XV.	a, l, s		$n = \frac{2s}{l+a}$	V.
XVI.	d, l, s		$n = \frac{2l+d}{2d} + \sqrt{\left(\frac{2l+d}{2d}\right)^2 - \frac{2s}{d}}$	VII.
XVII.	d, n, l	a	$a = l-(n-1)d$	I.
XVIII.	d, n, s		$a = \frac{s}{n} - \frac{(n-1)d}{2}$	VI.
XIX.	l, n, s		$a = \frac{2s}{n} - l$	V.
XX.	d, l, s		$a = \frac{1}{2}d + \sqrt{(l+\frac{1}{2}d)^2 - 2ds}$	VIII.

9. The following examples will enable the student to understand clearly the interpretation and application of these formulæ.

10. *To find the last term* of an arithmetical series when the first term, the common difference, and the number of terms are given:—

RULE.

$$l = a + (n-1)\,d. \quad \text{(I.)}$$

INTERPRETATION.—*The last term of a series is found by adding the first term to the product of the common difference by* 1 *less than the number of terms.*

EXAMPLE.—What is the tenth term of the arithmetical series 1, 3, 5, &c.?

OPERATION.

Here we have given the *first term* 1, the *common difference* 2, and the *number of terms* 10; to find the tenth or *last term.*

Then $l = a + (n-1)\,d = 1 + (10-1) \times 2 = 1 + 9 \times 2 = 1 + 18 = 19.$ *Ans.*

11. *To find the common difference* of an arithmetical series when the first term, the last term, and the number of terms are given:—

RULE.

$$d = \frac{l-a}{n-1}. \quad \text{(IX.)}$$

INTERPRETATION.—*To find the common difference of an arithmetical series,—Subtract the first term from the last term and divide the difference thus obtained by one less than the number of terms.*

EXAMPLE.—The first term of an arithmetical series is 3, the 13th term 55: find the common difference.

OPERATION.

Here we have given the *first term* 3, the *last term* 55, and the *number of terms* 13, to find the common difference.

Then $d = \frac{l-a}{n-1} = \frac{55-3}{13-1} = \frac{52}{12} = 4\tfrac{1}{3} =$ *Ans.*

12. *To find the sum* of an arithmetical series when the first term, the last term, and the number of terms are given:—

RULE.

$$s = (a + l)\,\frac{n}{2}. \quad \text{(V.)}$$

INTERPRETATION.—*Add the first and last terms together and multiply their sum by half the number of terms.*

EXAMPLE.—Find the sum of an arithmetical series whose first term is 2, last term 50, and number of terms 17.

OPERATION

Here we have given the first term 2, the last term 50 and the number of terms 17 to find s, the sum of the series.

Then $s = (a+l)\frac{n}{2} = (2+50)\times\frac{17}{2} = 52\times\frac{17}{2} = 26\times17 = 442$. *Ans.*

13. *To find the common difference* when the last term, the number of terms, and the sum of the series are given:

RULE.

$$d = \frac{2nl-2s}{n(n-1)}. \quad \text{(xii.)}$$

Interpretation.—*Take twice the product of the number of terms by the last term, and from it subtract twice the sum of the series. Divide the resulting difference by the product of the number of terms by 1 less than the number of terms and the quotient will be the common difference.*

Example.—In an arithmetical series the last term is 80, the number of terms 11, and the sum of the series 746, required the common difference.

OPERATION.

Here we have given l, n, and s to find d and since $l=80$, $n=11$, and $s=746$ we have:

$$d = \frac{2nl-2s}{n(n-1)} = \frac{(2\times11\times80)-(2\times746)}{11\times(11-1)} = \frac{1760-1492}{11\times10} = \frac{268}{110} = 2\tfrac{24}{55}.$$

14. *To find the number of terms* of an arithmetical series when the first term, the common difference, and the sum of the series are given:—

RULE.

$$n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2}. \quad \text{(xiv.)}$$

Interpretation.—*I. Subtract the common difference from twice the first term, divide the remainder by twice the common difference, square the quotient, add the result to the quotient obtained by dividing twice the sum of the series by the common difference and extract the square root of this sum.*

II. Next, from the common difference subtract twice the first term, divide the remainder by twice the common difference, and to the quotient add the square root obtained in I. The sum will be the number of terms.

Example.—The first term of an arithmetical progression is 7, the common difference $\frac{1}{4}$, and the sum of all the terms 142. What is the number of terms?

OPERATION.

Here we have given a, d, and s, to find n, and since $a = 7$, $d = \frac{1}{4}$, and $s = 142$, we have

$$n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2} = \frac{\frac{1}{4}-2\times 7}{2\times\frac{1}{4}} + \sqrt{\frac{142\times 2}{\frac{1}{4}} + \left(\frac{2\times 7-\frac{1}{4}}{2\times\frac{1}{4}}\right)^2} =$$

$$\frac{\frac{1}{4}-14}{\frac{1}{2}} + \sqrt{\frac{284}{\frac{1}{4}} + \left(\frac{14-\frac{1}{4}}{\frac{1}{2}}\right)^2} = -\frac{13\frac{3}{4}}{\frac{1}{2}} + \sqrt{1136+\left(\frac{13\frac{3}{4}}{\frac{1}{2}}\right)^2} = -27\tfrac{1}{2} +$$

$$\sqrt{1136 + (27\tfrac{1}{2})^2} = -27\tfrac{1}{2} + \sqrt{1136+756\tfrac{1}{4}} = -27\tfrac{1}{2} + \sqrt{1892\tfrac{1}{4}} = -27\tfrac{1}{2} + 43\tfrac{1}{2}$$

$= 16$. *Ans.*

Exercise 156.

1. In an arithmetical series the first term is 4, the number of terms 17, and the sum of the series 884. What is the last term? *Ans.* 100.
2. The extremes of an arithmetical series are 21 and 497, and the number of terms is 41. What is the common difference? *Ans.* $11\frac{9}{10}$.
3. In an arithmetical series the first term is 12, the last term 96, and the common difference is 6. Required the number of terms? *Ans.* 15.
4. In an arithmetical series the last term is 14, the common difference 1, and the sum of the series 105. Required the number of terms? *Ans.* 15.
5. The first term of an arithmetical series is ⅔, the common difference ⅔, and the sum of the series 1180. What is the last term? *Ans.* 39⅓.
6. If the extremes of an arithmetical series are 8 and 170, and the sum of the series 4895, what is the common difference? *Ans.* 3.
7. If the extremes of an arithmetical series are 5 and 27½, and the common difference 2¼, what is the number of terms? *Ans.* 11.
8. If the first term of a series is 2, the last term 478 and the number of terms 86, what is the sum of the series? *Ans.* 20640.
9. In an arithmetical series the last term is 998, the first term 2 and the common difference 6. What is the sum of the series? *Ans.* 83500.
10. In an arithmetical series the first term is 5, the number of terms 11 and the common difference 2¼. What is the last term? *Ans.* 27½.
11. In an arithmetical series the last term is 199, the common difference is 11 and the number of terms 19. Required the sum of the series? *Ans.* 1900.
12. The sum of an arithmetical series is 39840, and the extremes are 2 and 478. What is the number of terms? *Ans.* 166.
13. The sum of an arithmetical series is 83500, and the extremes are 998 and 2. Required the common difference? *Ans.* 6.

14. A snail crawls up a flag staff 150 feet high and upon reaching the top begins to descend. In what time will he again reach the ground if he goes 2 feet the first day, 4 feet the second, 6 feet the third, and so on?

Ans. 15 days, 15 hours, 10 min. 27·264 sec.

15. The sum of an arithmetical series is 83500, the first term is 2 and the common difference 6, what is the last term?

Ans. 998.

16. A person wishes to discharge a debt of $1125 in 18 annual payments which shall increase in arithmetical progression. How much must his first payment be in order that the last may be $120?

Ans. $5.

17. In an arithmetical series the extremes are 5 and 27½ and the number of terms is 11. What is the common difference?

Ans. 2¼.

18. 220 stones are placed in a straight line exactly 2½ yards apart, the first being 2½ yards from a basket, how far will a person go whilst picking up the stones, returning with one at a time and depositing it in the basket?

Ans. 69$\frac{1}{16}$ miles.

19. The sum of an arithmetical series is 39840, the number of terms is 166 and the last term is 478. What is the first term?

Ans. 2.

20. A person travelled from Toronto to Kingston, in 12 days, walking 4 miles the first day, 6 miles the second, 8 miles the third, and so on. How far is Toronto from Kingston?

Ans. 180 miles.

21. The clocks of Venice strike from 1 to 24. How many strokes does one of these clocks make in the day?

Ans. 300.

GEOMETRICAL PROGRESSION.

15. Quantities are said to be in Geometrical Progresion when they increase or decrease by a common multilier.

Thus 3, 12, 48, 192, &c., are in geometrical progression, the common ratio common multiplier being 4.

100, 20, 4, ⅘, $\frac{4}{25}$, &c., are in geometrical progression, the common ratio ing ⅕.

16. In *geometrical progression* there are five things to considered:

1. *The first term.*
2. *The last term.*
3. *The common ratio.*
4. *The number of terms.*
5. *The sum of the series.*

As in arithmetical progression, these five quantities are so related that any three of them being given the other two can be found, and hence there are 20 distinct cases arising from their combinations.

17. Representing these five quantities by letters, thus,

a = *the first term.*
l = *the last term.*
r = *the common ratio.*
n = *the number of terms.*
s = *the sum of the series.*

the general expression for a geometrical series becomes

$$a+ar+ar^2+ar^3+ar^4+ar^5+, \text{\&c.},$$

where the index of r is always *one less* than the number of the term.

Thus in the third term the index of r is 2, which is *one less* than the number of the term: in the fifth term the index of r is 4, which is *one less* than the number of the term, &c.

Hence $l = ar^{n-1}$; that is, the last term is equal to the first term multiplied by the common ratio raised to that power which is indicated by one less than the number of terms.

18. Since the sum of the series is equal to the sum of all the terms.

$s = a+ar+ar^2+ar^3+ \ldots\ldots ar^{n-3}+ar^{n-2}+ar^{n-1}$, multiplying by r we get
$sr = \quad ar+ar^2+ar^3+ \ldots\ldots ar^{n-3}+ar^{n-2}+ar^{n-1}+ar^n$.

Hence $sr-s = ar^n-a$; or $s(r-1) = a(r^n-1)$, and therefore $s = \dfrac{a(r^n-1)}{r-1}$

That is, the sum of the series is found by finding that power of the common ratio which is expressed by the number of terms, subtracting 1 from this, dividing the remainder by one less than the common ratio and multiplying the quotient by the first term.

NOTE.—The second of the above series is found from the first by multiplying both sides of the equation by r, and in subtracting we take the terms of the upper series from the corresponding terms of the lower. Only the first three or four and the last three or four terms are written and between ar^3 and ar^{n-3} there may be any number of intermediate terms. The ar^{n-3} in the lower series is obtained by multiplying the term before ar^{n-3} in the upper series, which is ar^{n-4}, by r.

19. From the formula obtained in Art. 17 we get by transposing the terms, &c.

$$l = ar^{n-1}$$

$$a = \frac{l}{r^{n-1}}$$

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

$$n = \frac{\log. l - \log. a}{\log. r} + 1$$

And substituting these values of l, a, r, n in the formula obtained in Art. 18 we find

$$s = \frac{rl-a}{r-1}$$

$$s = \frac{l(r^n-1)}{(r-1)r^{n-1}}$$

$$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$$

add these together with the two formulas obtained in Arts. 17 and 18,

$$s = \frac{a(r^n-1)}{r-1}$$

$$l = ar^{n-1}$$

are the fundamental formulas of geometrical progression from which the other fifteen are derived by reduction. Thus,

$s = \frac{rl-a}{r-1}$, *gives formulas for* s, r, l, *and* $a = 4$

$s = \frac{l(r^n-1)}{(r-1)r^{n-1}}$ " " s, r, l, *and* $n = 4$

$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$ " " s, l, n, *and* $a = 4$

$s = \frac{a(r^n-1)}{r-1}$ " " s, r, a, *and* $n = 4$

$l = ar^{n-1}$ " " l, a, r, *and* $n = 4$

Total, 20

20. The following table gives the 20 formulas for geometrical progression with their relations, &c. It will be observed that questions involving formulas III, XII, XIV, and XVI cannot be solved by common arithmetic, but require the aid of the higher mathematics. All the ormulas for n involve the use of logarithms.

No.	Given.	Required.	Formulas.	Whence derived.
I.	a, r, n	l	$l = ar^{n-1}$	fundamental.
II.	a, r, s		$l = \frac{a+(r-1)s}{r}$	VI.
III.	a, n, s		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$	VII.
IV.	r, n, s		$l = \frac{(r-1)sr^{n-1}}{r^n-1}$	VIII.
V.	a, r, n	s	$s = \frac{a(r^n-1)}{r-1}$	fundamental.
VI.	a, r, l		$s = \frac{rl-a}{r-1}$	V. and I.
VII.	a, n, l		$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$	V. and XIII.
VIII.	r, n, l		$s = \frac{l(r^n-1)}{(r-1)r^{n-1}}$	V. and IX.
IX.	r, n, l	a	$a = \frac{l}{r^{n-1}}$	I.
X.	r, n, s		$a = \frac{(r-1)s}{r^n-1}$	V.
XI.	r, l, s		$a = r(l-s)+s$	VI.
XII.	n, l, s		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$	VII.
XIII.	a, n, l	r	$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$	I.
XIV.	a, n, s		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$	V.
XV.	a, l, s		$r = \frac{s-a}{s-l}$	VI.
XVI.	n, l, s		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$	VIII.
XVII.	a, r, l	n	$n = \frac{\log. l - \log. a}{\log. r} + 1$	I.
XVIII.	a, r, s		$n = \frac{\log. [a + (r-1)s] - \log. a}{\log. r}$	V.
XIX.	a, l, s		$n = \frac{\log. l - \log. a}{\log. (s-a) - \log. (s-l)} + 1$	VII.
XX.	r, l, s		$n = \frac{\log. l - \log. [rl-(r-1)s]}{\log. r} + 1$	VIII.

APPLICATIONS.

21. Given the first term, the common ratio, and the number of terms, *to find the last term*:—

RULE.

$l = ar^{n-1}$. (I.)

INTERPRETATION.—*Multiply the first term by the common ratio raised to that power which is indicated by one less than the number of terms. The result will be the last term.*

EXAMPLE.—What is the 9th term of the series 7, 21, 63, &c.?

OPERATION.

Here $a = 7$, $r = 3$, and $n = 9$.

Then $l = ar^{n-1} = 7 \times 3^{9-1} = 7 \times 3 = 7 \times 6561 = 45927$. *Ans.*

22. Given the first term, the common ratio, and the last term, *to find the sum of the series*:—

RULE.

$$s = \frac{rl - a}{r - 1}. \quad \text{(VI.)}$$

INTERPRETATION.—*Subtract the first term from the product of the common ratio by the last term and divide the remainder by one less than the common ratio.*

EXAMPLE.—The first term of a geometrical series is 5, the common ratio 4, and the last term 1000000. What is the sum of all the terms?

OPERATION.

Here $a = 5$, $r = 4$, and $l = 1000000$.

Then $s = \frac{rl - a}{r - 1} = \frac{4 \times 1000000 - 5}{4 - 1} = \frac{3999995}{3} = 1333331\frac{2}{3}$. *Ans.*

23. Given the first term, the common ratio, and the number of terms, *to find the sum of the series*:—

RULE.

$$s = a\left(\frac{r^n - 1}{r - 1.}\right) \quad \text{(V.)}$$

INTERPRETATION.—*Find that power of the common ratio which is indicated by the number of terms, subtract one from it, and divide the remainder by one less than the common ratio.*

Lastly, multiply the quotient thus obtained by the first term of the series, and the result will be the sum of all the terms.

EXAMPLE.—The first term of a geometrical series is 3, the common ratio is 4, and the number of terms 9. Required the sum of the series.

Here $a=3$, $r=4$, and $n=9$.

Then $s=a\left(\frac{r^n-1}{r-1}\right)=3\times\frac{4^9-1}{4-1}=3\times\frac{262144-1}{3}=262143$. *Ans.*

24. *To find the common ratio* when the first term, the last term, and the sum of the terms are given:—

RULE.

$$r=\frac{s-a}{s-l} \text{ (XV.)}$$

INTERPRETATION.—*Divide the difference between the first term and the sum by the difference between the last term and the sum: the quotient will be the common ratio.*

EXAMPLE.—The first term of a geometrical series is 1, the last term 19683, and the sum of all the terms, 29524. What is the common ratio?

OPERATION.

Here $a=1$, $l=19683$, and $s=29524$.

Then $r=\frac{s-a}{s-l}=\frac{29524-1}{29524-19683}=\frac{29523}{9841}=3$. *Ans.*

EXERCISE 157.

1. A nobleman dying left 11 sons, to whom he bequeathed his property as follows: to the youngest he gave £1024; to the next, as much and a half: to the next 1½ of the preceding son's share; and so on. What was the eldest son's fortune; and what was the amount of the nobleman's property?
Ans. The eldest son received £59049, and the father was worth £175099.
2. The first term of a geometrical progression is 7, the last term is 1240029, and the sum of all the terms is 1860040. What is the ratio? *Ans.* 3
3. What debt can be discharged in a year by monthly payments in geometrical progression, the first term being £1, and the last £2048; and what will be the common ratio?
Ans. The debt will be £4095; and the ratio 2.
4. The ratio of the terms of a geometrical progression is $\frac{3}{2}$, the number of terms is 8, and the last term is $106\frac{403}{512}$. What is the sum of all the terms? *Ans.* $307\frac{441}{512}$.
5. In a geometrical progression the first term is 1, the number of terms 7. and the common ratio 3, what is the sum of the series?
Ans. 109

6. The first term of a geometrical progression is 1, the last term is 10077696, and the number of terms is 10. What is the sum of all the terms? *Ans.* 12093235.

7. The first term of a geometrical progression is 6, the last term is 3072, and the sum of all the terms is 6138. What is the ratio? *Ans.* 2.

8. The ratio of the terms of a geometrical progression is 2, the number of terms is 11, and the sum of all the terms is 20470. What is the last term? *Ans.* 10240.

9. A gentleman married his daughter on New Year's day, and gave her husband 1 shilling towards her portion, and was to double it on the first day of every month during the year. What was her portion? *Ans.* £204 15s.

10. What will be the price of a horse sold for 1 farthing for the first nail in his shoes, 2 farthings for the second, 4 for the third, &c., allowing 8 nails in each shoe? *Ans.* £4473924 5s. 3¾d.

11. The first term of a geometrical progression is 4, the last term is 78732, and the number of terms is 10. What is the ratio? *Ans.* 3.

12. A person travelling goes 5 miles the first day, 10 miles the second day, 20 miles the third day, and so on increasing in geometrical progression. If he continue to travel in this way for 7 days, how far will he go the last day? *Ans.* 320 miles.

13. The first term of a geometrical progression is 5, the last term is 327680, and the ratio is 4. What is the sum of all the terms? *Ans.* 436905.

14. A king in India, named Sheran, wished (according to the Arabic author Asephad) that Sessa, the inventor of chess, should himself choose a reward. He requested the king to give him 1 grain of wheat for the first square, 2 grains for the second square, 4 grains for the third square, and so on; reckoning for each of the 64 squares of the board twice as many grains as for the preceding. Sheran was angry at a demand apparently so insignificant; but when it was calculated, to his astonishment it was found to be an enormous quantity. What was the number of grains of wheat, and what was its worth at $1·50 per bushel, reckoning 7680 grains to a pint?

Ans. 18446744073709551615 grains.
37529996894754 bushels.
$56294995342131.

15. The ratio of the terms of a geometrical progression is 3, the number of terms is 10, and the sum of all the terms is 295240. What is the last term? *Ans.* 196830.

16. The first term of a geometrical progression is 1, the last term is 2048, and the number of terms is 12. What is the sum of all the terms? *Ans.* 4095.

17. The first term of a geometrical progression is 5, the ratio is 4, and the number of terms 9. What is the last term?

Ans. 327680.

25. When the common ratio of a geometrical series is a proper fraction, i. e., less than 1, the series is a descending one, and when the number of terms becomes very large r^n becomes very small. In an infinite descending series r^n becomes infinitely small, i. e., its value becomes $= 0$, and therefore ar^n may be neglected, and the formula for finding the sum becomes $s = \frac{ar^n - a}{r-1} = \frac{-a}{r-1} = \frac{a}{1-r}$. Hence for finding the sum of any *infinite* series when r is less than 1:—

RULE.

$$s = \frac{a}{1-r} \text{ (xxi.)}$$

INTERPRETATION.—*The sum of an infinite series is found by dividing the first term by unity minus the common ratio.*

EXAMPLE 1.—What is the sum of the infinite series $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125}$, &c.?

OPERATION.

Here $a = 1$ and $r = \frac{1}{5}$

Then $s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4} = 1\frac{1}{4}$. *Ans.*

EXAMPLE 2.—What is the sum of the infinite series $\cdot\dot{7}8\dot{4}$?

OPERATION.

Here $a = \frac{784}{1000}$ and $r = \frac{1}{1000}$.

Then $s = \frac{a}{1-r} = \frac{\frac{784}{1000}}{1-\frac{1}{1000}} = \frac{\frac{784}{1000}}{\frac{999}{1000}} = \frac{784}{999}$. *Ans.*

EXERCISE 158.

1. What is the sum of the infinite series $\frac{3}{7}$, $\frac{6}{35}$, $\frac{12}{175}$, &c.? *Ans.* $\frac{5}{7}$.
2. What is the sum of the infinite series 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c.? *Ans.* 8.
3. What is the sum of the infinite series $\cdot\dot{7}\dot{9}$? *Ans.* $\frac{79}{99}$.
4. What is the sum of the infinite series $\cdot\dot{1}23\dot{4}$? *Ans.* $\frac{1234}{9999}$.

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26. To insert any number of means between two given extremes:

RULE.

If the series is an arithmetical one, find the common difference by formula IX. ART. 8. *Then add this common difference to the first term and the result will be the second term; add the common difference to the second and the result will be the third term, &c.*

If the series is a geometrical one, find the common ratio by formula XIII. ART. 20. *Then multiply the first term by the common ratio and the product will be the second term; multiply the second term by the common ratio and the result will be the third, &c.*

EXAMPLE 1.—Insert 7 arithmetical means between 3 and 51.

OPERATION.

Since there are 7 means and 2 extremes the number of terms is 9.

Then $d=\frac{l-a}{n-1}=\frac{51-3}{9-1}=\frac{48}{8}=6.$

1st term=3; 2nd=3+6=9; 3rd=9+6=15; 4th=15+6=21; 5th=21+6=27; 6th=27+6=33, and so on.

And series is 3, 9, 15, 21, 27, 33, 39, 45, 51.

EXAMPLE 2.—Insert 6 geometrical means between 1 and 128.

OPERATION.

Since there are 6 means and 2 extremes the number of terms is 8.

Then $r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}=\left(\frac{128}{1}\right)^{\frac{1}{8-1}}=(128)^{\frac{1}{7}}=2.$

Hence 2nd term $=1\times2=2$; 3rd term $=2\times2=4$; 4th $=4\times2=8$, &c.

And series is 1, 2, 4, 8, 16, 32, 64, 128.

EXERCISE 159.

1. Insert 9 arithmetical means between 2 and 92.
Ans. 2, 11, 20, 29, 38, 47, 56, 65, 74, 83, 92.
2. Insert 4 arithmetical means between 7 and 50.
Ans. 7, $15\frac{3}{5}$, $24\frac{1}{5}$, $32\frac{4}{5}$, $41\frac{2}{5}$, 50.
3. Find 8 geometrical means between 4096 and 8.
Ans. 2048, 1024, 512, 256, 128, 64, 32, and 16.
4. Find 7 geometrical means between 14 and 23514624.
Ans. 84, 504, 3024, 18144, 108864, 653184, and 3919104.

POSITION.

27. Position is a rule which enables us to solve, by means of assumed numbers, a class of problems which we could not otherwise solve without the aid of algebra.

NOTE.—Position is also called the Rule of False, or the Rule of Trial and Error.

28. Position is divided into:—

1st. Single Position—when only one assumed number is used.

2nd. Double Position—when two assumed numbers are used.

29. Single position is employed in the solution of those problems in which the required number is increased or decreased in any given ratio, i. e., when it is increased or diminished by *any part of itself*, or when it is *multiplied or divided by any given number.*

30. Double Position is employed in the solution of those problems in which the *result* found by increasing or decreasing the required number in any given ratio, is itself increased or diminished by some other number which is no known part or multiple of the required number.

SINGLE POSITION.

31. Single Position proceeds upon the principle that the results are proportional to the numbers used, and is employed in all cases when the problem can be stated algebraically in the form of $ax=b$, where $x=$ the required number, a the given multiplier, integral or fractional, and b the given result.

32. Let it be required to find a value of x such that $ax=b$. Suppose x' to be this value, and instead of b we obtain b' for the result. Then we have $ax=b$ and $ax'=b'$, and dividing we get $\frac{ax'}{ax}=\frac{b'}{b}$ or $\frac{x'}{x}=\frac{b'}{b}$ whence $b' : b :: x' : x$ or $x=\frac{b}{b'}\times x'$.

Hence for single position we deduce the following

RULE.

Assume a number, and perform with it the operations described in the question; then say, as the result obtained is to the number used, so is the true or given result to the number required.

Example 1.—What number is that which being increased by its fourth part and diminished by its fifth part gives 63 for the result?

OPERATION

Assume any number, 40.* Then *one-fourth* of number $=10$, and *one-fifth* $=8$.

---

* For the sake of convenience we assume a number of which we can take the required parts without using fractions.

40+10−8=42, which by the question should have been 63.

Then—Result obtained : Result required : : Number used : Number required.

Or, 42 : 63 : : 40 : $\frac{63\times40}{42}$=60. *Ans.*

PROOF.—60+¼ of 60−⅕ of 60=63.

EXAMPLE 2.—A teacher being asked how many pupils he had, replied, if you add ⅓, ¼, and ⅙ of the number together, the sum will be 18; what was their number?

OPERATION.

Assume 60 to be the number of pupils.
Then one-third of 60=20
one-fourth of 60=15
one-sixth of 60=10
Sum=45, but it should, by question, equal 18.

Then 45 : 18 : : 60 : $\frac{18\times60}{45}$=24. *Ans.*

PROOF.—⅓ of 24+¼ of 24+⅙ of 24=18.

EXERCISE* 180.

1. A gentleman distributed 78 pence among a number of poor persons, consisting of men, women, and children; to each man he gave 6d., to each woman 4d., to each child 2d.; there were twice as many women as men, and three times as many children as women. How many were there of each?
Ans. 3 men, 6 women, and 18 children.

2. A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness. What did he give for each?
Ans. He gave for the harness, £6 13s. 4d.; for the horse, £13 6s. 8d.; and for the chaise, £40.

3. A's age is double that of B's; B's is treble that of C's; and the sum of all their ages is 140. What is the age of each?
Ans. A's is 84, B's 42, and C's 14.

4. After paying away ¼ of my money; and then ⅕ of the remainder, I had 72 guineas left. What had I at first? *Ans.* 120 guineas.

---

* All questions in position may be solved by simple analysis, and very frequently this is the better method, and indeed the teacher should insist upon the pupil thus solving each problem. The following will serve as examples of the mode of solution.

EXAMPLE 3.—Since 140 is equal to A's age, + B's age, + C's age, and B's age is equal to three times C's, and A's to 6 times C's, it follows that 140 is equal to 1+3+6=10 times C's age, and hence C's age is $\frac{1}{10}$ of 140=14; B's=14×3=42; and A's=14×6=84.

5. A can do a piece of work in seven days; B can do the same in 5 days; and C in 6 days. In what time will all of them execute it? *Ans.* In $1\frac{103}{107}$ days.
6. A and B can do a piece of work in 10 days; A by himself can do it in 15 days. In what time will B do it? *Ans.* In 30 days.
7. A cistern has three pipes; when the first is opened all the water runs out in one hour; when the second is opened, it runs out in two hours; and when the third is opened, in three hours. In what time will it run out, if all the pipes are kept open together? *Ans.* In $\frac{6}{11}$ hours.
8. What is that number whose $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{7}$ parts, taken together, make 27? *Ans.* 42.
9. There are 5 mills; the first grinds 7 bushels of corn in 1 hour, the second 5 in the same time, the third 4, the fourth 3, and the fifth 1. In what time will the five grind 500 bushels if they work together? *Ans.* In 25 hours.
10. There is a cistern which can be filled by a pipe in 12 hours; it has another pipe in the bottom, by which it can be emptied in 18 hours. In what time will it be filled, if both are left open? *Ans.* In 36 hours.

DOUBLE POSITION.

33. When the number sought is to be increased or diminished by some *absolute* number, which is not a known multiple, or part of it—or when *two* propositions, neither of which can be banished, are contained in the problem, we use *double* position, assuming *two* numbers. If the number sought is, during the process indicated by the question, to be involved or evolved, we obtain only an *approximation* to the quantity required. In other words *double* position is employed in all cases in which the problem stated algebraically would take the form of

$$ax + b = c$$

where x is the number sought, a the given multiplier, integral or fractional, b the given increment, and c the given result.

Example 7. By Analysis.—Since A can do the whole work in 7 days, in 1 day he will do $\frac{1}{7}$ of the whole work, similarly in one day B will do $\frac{1}{5}$, and C $\frac{1}{6}$ of the whole work. Therefore working together they will do $\frac{1}{7}+\frac{1}{5}+\frac{1}{6}=\frac{107}{210}$ of the whole work, and they will require as many days to do the whole work as $\frac{107}{210}$ is contained times in 1, i. e., $1\div\frac{107}{210}=1\frac{103}{107}$ days. *Ans.*

34. Let it be required to find a value for x such as to satisfy the equation, $ax + b = c$.

In such a case assume any two known numbers n and n' and perform on these the operations indicated in the question, and let the errors in the result be e and e', both suppose in excess.

Then $an + b = c + e$ (I) and $an' + b = c + e'$ (II), and, by the question, $ax + b = c$ (III).

Subtracting III from I we get $an - ax = e$, or $a(n - x) = e$ (IV).

Subtracting III from II we get $an' - ax = e'$, or $a(n' - x) = e'$ (V).

Dividing IV by V we get $\frac{a(n-x)}{a(n'-x)} = \frac{e}{e'}$ or $\frac{n-x}{n'-x} = \frac{e}{e'}$.

And reducing this we get $x = \frac{n'e - ne'}{e - e'}$.

Hence for double position we deduce the following :—

RULE.

I. Assume two convenient numbers, and perform upon them the processes supposed by the question, marking the error derived from each with + or —, according as it is an error of excess, or of defect.

II. Multiply each assumed number into the error which belongs to the other; and, if the errors are both plus, or both minus, divide the difference of the products by the difference of the errors. But, if one is a plus, and the other is a minus error, divide the sum of the products by the sum of the errors. In either case, the result will be the number sought, or an approximation to it.

EXAMPLE 1.—There is a fish whose head is 8 feet long, his tail is as long as his head and half his body, and his body is as long as his head and tail; what is the whole length of the fish?

OPERATION.

Assume 24 feet as the length of body.
Then tail $= 8 + \frac{1}{2}$ of $24 = 8 + 12 = 20$
Body = head + tail $= 8 + 20 = 28$
Assumed length of body $= 24$
Error $= +4$

Assume 28 feet for length of body.
Then tail $= 8 + \frac{1}{2}$ of $28 = 8 + 14 = 22$
Body = head + tail $= 8 + 22 = 30$
Assumed length of body $= 28$
Error $= +2$

Errors.		Assumed numbers.		Products.
+ 4	×	28	=	112
+ 2	×	24	=	48
Difference of errors = 2			difference of products =	64

Then $64 \div 2 = 32 =$ length of body
$8 + \frac{1}{2}$ of $32 = 8 + 16 = 24 =$ " tail
$8 =$ " head
$64 =$ length of fish.

EXAMPLE 2.—A laborer contracted to work 80 days for 75 cents per day, and to forfeit 50 cents for every day he should be idle during that time. He received $25; now how many days did he work, and how many days was he idle?

OPERATION.

Suppose he worked 50 days, then he was idle 30 days.

Sum earned = 50 × 75 = $37·50 | True result = $25·00
Sum forfeited = 30 × 50 = 15·00 | Result obtained = 22·50
Sum received = 22·50 | Error = − 2·50

Again: suppose he worked 40 days; then he lost 40 days.

Sum earned = 40 × 75 = $30·00 | Result required = $25·00
Sum forfeited = 40 × 50 = 20·00 | Result obtained = 10·00
Sum received = 10·00 | Error = − 15·00

Errors.		Assumed numbers.		Products.
− 15	×	50	=	750
− 2½	×	40	=	100

Difference of errors = 12½. Difference of products = 650.

Therefore result required = 650 ÷ 12½ = 52 days.
Number of idle days = 80 − 52 = 28. *Ans.*
Proof.—Sum earned = 52 × 75 = $39·00
Sum forfeited = 28 × 50 = 14·00
Sum received = $25·00

Example 3.—What number is that which being multiplied by 3, the product increased by 4, and that sum divided by 8, the quotient shall be 32?

OPERATION.

Assume 40 to be the number.

Then 40 × 3 = 120 + 4 = 124 ÷ 8 = 15½ = result obtained.
32 = result required.
Error = − 16½

Again: assume 100 to be the number.

Then 100 × 3 = 300 + 4 = 304 ÷ 8 = 38 = result obtained.
32 = result required.
Error = + 6

Errors.		Assumed numbers.	
− 16½	×	100	= 1650
+ 6	×	40	= 240
Sum of error = 22½			Sum of products = 1890

$$\text{Required number} = \frac{1890}{22\frac{1}{2}} = 84. \ \textit{Ans.}$$

Proof.—84 × 3 = 252 + 4 = 256 ÷ 8 = 32.

Note.—In this example we take the sum of the errors for a divisor and the sum of the products for a dividend, because the errors are not *both plus* or *both minus.*

EXAMPLE.—What is that number which is equal to 4 times its square root + 21 ?

OPERATION.

Assume 64
√64=8
4
32
21
53, result obtained.
64, result required.
−11, difference.
81
891

Assume 81
√81=9
4
36
21
57, result obtained.
81, result required.
−24, difference.
64
1536
891
13)645

The first approximation is 49·6154

It is evident that 11 and 24 are not the errors in the assumed numbers multiplied or divided by the same quantity, and, therefore, as the reason upon which the rule is founded, does not apply, we obtain only an approximation. Substituting this, however, for one of the assumed numbers, we obtain a still nearer approximation.

SECOND RULE.

Find the errors by the last rule; then divide their difference (if they are both of the same kind), or their sum (if they are of different kinds), into the product of the difference of the numbers and one of the errors. The quotient will be the correction of that error which has been used as multiplier.

NOTE.—This rule depends upon the principle that the difference between the assumed numbers and the true numbers is proportional to the differences of the results obtained using the assumed numbers and that given in the problem. As in the last rule, when the question could not by algebra be resolved by an equation of the first degree, the rule gives only an approximation to the correct result.

EXAMPLE.—If to four times the price of my horse £10 be added, the result will be £100. What is the price of my horse ?

OPERATION.

Assume £19, and secondly £25 as the price of the horse—

Then 19
4
76
10
86, the result obtained.
100, the result required.
−14 is an error of *defect*.

25
4
100
10
110, the result obtained.
100, the result required.
+10 is an error of *excess*.

The errors are of *different* kinds: and their *sum* is 14+10=24; and the difference of the assumed numbers is 25−19=6. Therefore

```
      14, one of the errors,
multiplied by  6, the difference of the numbers.   Then divide by
              ——
           24)84
              ——
```

and 3·5 is the correction for 19, the number which gave an error of 14.

19+(the error being one of *defect*, the correction is to be *added*) 3·5=22·5 =£22 10s. is the required quantity.

EXERCISE 161.

1. A son asked his father how old he was, and received the following answer: Your age is now $\frac{1}{4}$ of mine, but 5 years ago it was only $\frac{1}{5}$. What are their ages? *Ans.* 80 and 20.
2. Required what number it is from which if 34 be taken, 3 times the remainder will exceed it by $\frac{1}{4}$ of itself? *Ans.* $58\frac{2}{7}$.
3. A and B go out of a town by the same road. A goes 8 miles each day; B goes 1 mile the first day, 2 the second, 3 the third, &c. When will B overtake A?

```
            A.    B.                     A.    B.
Suppose     5     1         Suppose      7     1
            8     2                      8     2
           ——     3                     ——     3
           40     4                     56     4
           15     5                     28     5
           ——    ——                     ——     6
         5)25    15                   7)28     7
           ——                           ——    ——
           −5                           −4    28
            7                            5
           ——                           ——
           35                           20
           20
           ——           5−4=1=difference of errors.
         1)15
```

We divide the entire error by the number of days in each case, which gives the error in one day.

4. What are those numbers which, when added, make 25; but when one is halved and the other doubled, give equal results. *Ans.* 20 and 5.
5. Two contractors, A and B, are each to build a wall of equal dimensions; A employs as many men as finish $22\frac{1}{2}$ perches in a day; B employs the first day as many as finish 6 perches, the second as many as finish 9, the third as many as finish 12, &c. In what time will they have built an equal number of perches? *Ans.* 12 days.
6. What is the number whose $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ multiplied together, make 24?

Suppose 12

½ = 6
¼ = 3

Product = 18
⅜ = 4½

81, result obtained.
24, result required.

+ 57, error,
64, the cube of 4.

3648, product.

57 + 21 = 78

Suppose 4

½ = 2
¼ = 1

Product = 2
⅜ = 1½

3, result obtained.
24, result required.

− 21, error.
1728, the cube of 12.

36288 to this product
3648 is added.

78)39936 is the sum,

And 512 the quotient.

$\sqrt[3]{512} = 8$, is the required number.

We multiply the alternate error by the *cube* of the supposed number, because the error belongs to $\frac{3}{64}$ part of the *cube* of the assumed numbers and not to the numbers themselves; for in reality it is the cube of some number that is required—since 8 being assumed, according to the question we have

$$\frac{8}{2} \times \frac{8}{4} \times \frac{3 \times 8}{8} = 24; \text{ or } \frac{3}{64} \times 8^3 = 24.$$

7. What number is it whose ½, ¼, ⅕, and ⅙, multiplied together, will produce 6998⅖? *Ans.* 36.

8. A said to B, give me one of your shillings and I shall have twice as many as you will have left. B answered, if you give me one shilling I shall have as many as you. How many had each? *Ans.* A 7, and B 5.

9. There are two numbers which, when added together, make 30; but the ½, ⅓, and ⅙ of the greater are equal to ½, ¾, ¼ of the lesser. What are they? *Ans.* 12 and 18.

10. A gentleman has 2 horses, and a saddle worth £50. The saddle, if set on the back of the first horse, will make his value double that of the second; but if set on the back of the second horse, will make his value treble that of the first. What is the value of each horse? *Ans.* £30 and £40.

11. A gentleman finding several beggars at his door, gave to each 4d. and had 6d. left, but if he had given 6d. to each, he would have 12d. too little. How many beggars were there? *Ans.* 9.

COMPOUND INTEREST.

35. Let P = the principal, I = the interest, A = the amount, t = the number of payments, and r = the rate per unit for one payment.

Then since r is the interest of $1 for one payment, the amount of $1 for one payment is $1+r$, and since the principal is always proportional to the amount:

$1 : 1+r :: P : P(1+r)$ = Amount of P at end of 1st period.
$1 : 1+r :: P(1+r) : P(1+r)^2$ = Amount of P at end of 2nd period.
$1 : 1+r :: P(1+r)^2 : P(1+r)^3$ = Amount of P at end of 3rd period.
$1 : 1+r :: P(1+r)^3 : P(1+r)^4$ = Amount of P at end of 4th period.

And so on; hence at the end of the t^{th} period $A = P(1+r)^t$, which is formula (I) in the margin.

$$A = P(1+r)^t \quad \text{(I)}$$

Dividing each side of (I) by $(1+r)^t$ we get formula (II) in the margin.

$$P = \frac{A}{(1+r)^t} \quad \text{(II)}$$

Dividing each side of (I) by P we get $(1+r)^t = \frac{A}{P}$; extracting the t root, and transposing the 1, we get formula (III).

$$r = \sqrt[t]{\frac{A}{P}} - 1 \quad \text{(III)}$$

Obtaining as before $(1+r)^t = \frac{A}{P}$ and applying the principle of logarithms we get log. $(1+r) \times t$ = log. A − log. P, and dividing each side by log. $(1+r)$ we get $t = \frac{\log. A - \log. P}{\log. (1+r)}$, which is (IV) of the margin.

$$t = \frac{\log. A - \log. P}{\log. (1+r)} \quad \text{(IV)}$$

Lastly, to find the time in which any sum of money will amount to n times itself at a given rate per cent. compound interest, we substitute nP for A in formula (I), which gives us $nP = P(1+r)^t$ and dividing each of these by P we get $n = (1+r)^t$ whence log. n = log. $(1+r) \times t$; or $t = \frac{\log. n}{\log. (1+r)}$, which is formula (V).

$$t = \frac{\log. n}{\log. (1+r)} \quad \text{(V)}$$

APPLICATIONS.

When the principal, rate per cent., and time are given *to find the amount:—*

RULE.

$$A = P(1+r)^t \text{ or } \log. A = \log. P + \log. (1+r) \times t. \quad (1)$$

INTERPRETATION.—*Multiply the logarithm of the amount of $1 for one payment by the number of payments, and to the product add the logarithm of the principal; the result will be the logarithm of the amount.*

II. Find the natural number corresponding to this logarithm and the result will be the answer.

EXAMPLE.—To what sum will $750 amount in 3 years, at 2 per cent., quarterly compound interest?

OPERATION.

Here P = 750, r = ·02, and t = 12, since there are 12 quarters in 3 years.
Then $A = P(1+r)^t$ or log. A = log. P + log. $(1+r) \times t$ = 2·875061 + 0·008600 × 12 = 2·978261 = log. of Answer. Hence amount = $951·17.

36. When the amount, rate, and time are given *to find the principal* :—

RULE.

$$P = \frac{A}{(1+r)^t}\,;\ or\ log.\ P = log.\ A - log.\ (1+r) \times t. \quad \text{(II.)}$$

INTERPRETATION.—*Take the number expressing the amount of $1 for one payment, and raise it to the power indicated by the number of payments.*

II. Divide the given amount by the number thus obtained and the quotient will be the required principal.

BY LOGARITHMS.

Take the logarithm of the amount of $1 for one payment, and multiply it by the number of payments.

Subtract the logarithm thus obtained from the logarithm of the given amount; the remainder will be the logarithm of the required principal.

EXAMPLE.—What principal put out at compound interest, at the rate of 3½ per cent. half-yearly, will amount to $8764·00 in 11 years?

OPERATION.

Here $A = 8764$, $r = ·035$ and $t = 22$.

Then $P = \dfrac{A}{(1+r)^t}$ or $\log. P = \log. A - \log. (1+r) \times t$.

$\log. P = 3·942702 - 0·014940 \times 22 = 3·942702 - 0·328680 = 3·614022$.

Hence P = $4111·70. *Ans.*

37. When the amount, principal, and time are given *to find the rate per cent.* :—

RULE.

$$r = \sqrt[t]{\left(\frac{A}{P}\right)} - 1\,;\ or\ log.\ (1+r) = \frac{log.\ A - log.\ P}{t} \quad \text{(III.)}$$

INTERPRETATION.—*Divide the amount by the principal, and extract that root of the quotient which is indicated by the number of payments.*

II. Subtract 1 *from the root thus obtained and the remainder will be the rate per unit, multiply this by* 100, *and the result will be the rate per cent.*

BY LOGARITHMS.

Subtract the logarithm of the principal from the logarithm of the given amount, and divide the difference by the number of payments; the result will be the logarithm of the amount of $1 for one payment.

Find the natural number corresponding to this, and from it subtract 1, the result will be the rate per unit, and this multiplied by 100 *gives the rate per cent.*

EXAMPLE.—At what rate per cent. compound interest, payable half-yearly, will $278 amount to $6742 in 27 years?

OPERATION.

Here $A = 6742$, $P = 278$ and $t = 54$.

Then $\log.\,(1 + r) = \dfrac{\log.\,A - \log.\,P}{t} = \dfrac{3{\cdot}828789 - 2{\cdot}444045}{54} = \dfrac{1{\cdot}384744}{54}$ $= {\cdot}0256434$. Hence $1 + r = 1{\cdot}06$, $r = {\cdot}06$, and rate per cent. $= 6$. *Ans.*

38. When the amount, principal, and rate are given *to find the time*:—

RULE.

$$t = \frac{\log.\,A - \log.\,P}{\log.\,(1 + r)} \quad \text{(IV.)}$$

INTERPRETATION.—*Subtract the logarithm of the principal from the logarithm of the given amount, and divide the remainder by the logarithm of the amount of $1 for one payment; the quotient will be the number of the payments.*

EXAMPLE.—In what time will $729 amount to $7143 at 2½ per cent. compound interest, quarterly?

OPERATION.

Here $A = 7143$, $P = 729$ and $r = {\cdot}025$.

Then $t = \dfrac{\log.\,A - \log.\,P}{\log.\,(1 + r)} = \dfrac{3{\cdot}853881 - 2{\cdot}862728}{0{\cdot}010724} = \dfrac{0{\cdot}991153}{0{\cdot}010724} = 92{\cdot}42$ payments $= 23{\cdot}105$ years $= 23$ years 1 month 7·8 days. *Ans.*

39. *To find in what time* any sum of money will amount to *n* times itself at any given rate per cent. compound interest:—

RULE.

$$t = \frac{\log.\,n}{\log.\,(1 + r)} \quad \text{(V.)}$$

INTERPRETATION.—*Find the logarithm of the number expressing to how many times itself the given sum is to amount, and divide it by the logarithm of the amount of $1 for one payment; the result will be the required time.*

EXAMPLE 1.—In what time will any sum of money amount to *five* times itself at 5 per cent. per annum, compound interest?

OPERATION.

Here $n = 5$ and $r = {\cdot}05$.

Then $t = \dfrac{\log.\,n}{\log.\,(1 + r)} = \dfrac{0{\cdot}698970}{0{\cdot}021189} = 32{\cdot}987$ yrs. $= 32$ years 11 months 25 days. *Ans.*

EXAMPLE 2.—In what time will any sum of money amount to *nine* times itself at 3½ per cent. quarterly, compound interest?

Here $n=9$ and $r=\cdot085$.

Then $t=\dfrac{\log. n}{\log. (1+r)}=\dfrac{0\cdot954243}{0\cdot014940}=63\cdot8716$ payments $=15\cdot9679$ years $=15$ years 11 months 18 days. *Ans.*

Exercise 162.

1. What is the amount and compound interest of $718·29 for 7 years at 4½ per cent. half yearly?
Ans. Amount=$1820·96.
Compound interest=$ 607·67.

2. In what time will any sum of money amount to seven times itself at 1½ per cent. quarterly, compound interest?
Ans. 32 years 8 months 2 days.

3. In what time will $111·11 amount to $1111·11 at 8 per cent. per annum, compound interest?
Ans. 29 years 11 months.

4. At what rate per cent. quarterly will $222·22 amount to $3333·33 in 30 years, compound interest being allowed?
Ans. $2\frac{7}{25}$.

5. In what time will any sum of money double itself at 7 per cent. per annum, compound interest?
Ans. 10 years 2 months 28 days.

6. What principal put out at compound interest at the rate of 2¼ per cent. quarterly will amount to $100 in 7 years?
Ans. $53·63.

7. To what sum will $2468·13 amount in 13 years at compound interest 3¾ per cent. half yearly?
Ans. $6427·705.

8. What principal will amount to $7187·40 in 11 years, compound interest at the rate of 4¼ per cent. half yearly being allowed?
Ans. $2856·723.

9. In what time will any sum of money amount to 19 times itself at 5¼ per cent. half yearly, compound interest?
Ans. 28 years 9 months 8 days.

ANNUITIES.

40. An Annuity is any periodical income payable at equal intervals, as yearly, half yearly, quarterly, &c.

41. An Annuity *in possession* is one that is entered upon already.

42. An Annuity *in reversion* or a *deferred annuity* is one whose first payment is not to be made until after the expiration of a given time or until the occurrence of a specified event.

43. An Annuity *certain* is one that is to continue for a fixed number of years.

44. An Annuity *contingent* or a *life annuity* is one that is to continue to be paid only so long as one or more individuals shall live.

45. A *Perpetuity* is an annuity that is to continue for ever.

46. An Annuity is in *arrears* when one or more payments are retained after they have become due.

47. The *amount of an annuity* is the sum of the payments forborne (i. e. in arrears) and the whole interest due upon them.

48. The *present worth of an annuity* is that sum which, being put out at interest until the annuity ceases, would produce a sum equal to what would have been accumulated had the annuity been left unpaid until that time.

49. Annuities are calculated at both simple and compound interest.

ANNUITIES AT SIMPLE INTEREST.

50. Let $a=$ a single payment of the annuity, $t=$number of payments, $r=$ rate per unit for one period, and $A=$amount of the annuity.

Then when the annuity is forborne any number of payments, the last payment being made at the time it falls due, is equal to a; last payment but one $=a+$interest on a for one period$=a+ar$; last but two$=a+$interest on a for two payments$=a+2ar$; last but three$=a+3ar$; last but four$=a+4ar$, &c.; and hence the first payment$=a+$interest on a for one less than the number of payments$=a+(t-1)\,ar$.

Hence the payments forborne, with their interest, constitute a series in arithmetical progression where the first term is a, the last term $a+(t-1)\,ar$, the common difference ar, the sum of the series A, and the number of terms t.

Then (Art. 5) $A=a+(a+ar)+(a+2ar)+(a+3ar)$, &c. $+\left\{a+(t-1)ar\right\}$

Whence (Art. 6) $A=\left\{a+a+(t-1)ar\right\}\frac{t}{2}=\left(1+\frac{(t-1)r}{2}\right)ta$, which is formula I in the margin.

$$A=at\left(1+\frac{(t-1)r}{2}\right)\ \text{(I.)}$$

$$a=\frac{2A}{t\left(2+(t-1)\right)r}\ \text{(II.)}$$

$$r=\frac{2(A-at)}{at(t-1)}\ \text{(III.)}$$

$$t=\frac{\sqrt{\left\{\frac{8rA}{a}+(2-r)^2\right\}}-(2-r)}{2r}\ \text{(IV.)}$$

Formulas II. III, and IV, are derived from formula I, by transposition, &c.

No general formula has yet been discovered for the summation of a series for finding the *present value* of an annuity at simple interest. The rule generally adopted for finding the present value of an annuity at simple interest is the following:—

Find the present worth of each payment by itself, discounting from the time it falls due—the sum of the present worth of all the payments will be the present worth of the annuity.

NOTE.—The absolute absurdity of purchasing annuities by simple interest is evident from the fact that the interest of the sum required to purchase an annuity, discounting at 5 per cent. simple interest, actually exceeds the annuity; i. e., to purchase an annuity to continue only a limited number of years, requires a sum which will yield a larger yearly interest for *ever*. Hence the various rules given for finding the present value of annuities at simple interest are, in effect, valueless.

APPLICATIONS.

51. When the annuity, number of payments forborne, and the rate per cent. of interest are given, to find the amount:—

RULE.

$$A = at\left\{(1 + \frac{(t-1)r}{2}\right\} \quad \text{(I.)}$$

INTERPRETATION.—*Multiply the rate per unit by one less than the number of payments and to half the result add* 1.

Multiply the number thus obtained by the product of the annuity by the number of payments, and the result will be the required amount.

EXAMPLE.—If a pension of \$600 per annum be forborne 5 years, to what sum will it amount at 4 per cent. simple interest?

OPERATION.

Here $a = 600$, $t = 5$, $r = {\cdot}04$.

Then $A = at\left\{1 + \frac{(t-1)r}{2}\right\} = 600 \times 5\left\{1 + \frac{(5-1) \times {\cdot}04}{2}\right\} = 3000 \times (1 + {\cdot}08) = 3000 \times 1{\cdot}08 = \$3240$. *Ans.*

52 When the amount of the annuity forborne, the number of payments forborne, and the rate per cent. of interest allowed, are given, *to find the annuity*:—

RULE.

$$a = \frac{2A}{t\,\{2 + (t-1)r\}} \quad \text{(II.)}$$

INTERPRETATION.—*Multiply the rate per unit by one less than the number of payments, and to the product add* 2.

Multiply this sum by the number of payments, and divide twice the given amount of the annuity by the product thus obtained; the result will be the annuity required.

EXAMPLE.—What annuity, payable quarterly, will amount to \$3225·25 in 7 years, at $4\frac{1}{2}$ per cent. per annum, simple interest?

OPERATION.

Here since the rate is $4\frac{1}{2}$ per cent. per annum, or ·045 per unit per annum, the rate per quarter $= \cdot 045 \div 4 = \cdot 01125$.

Then $t = 28$, $A = \$3225 \cdot 25$ and $r = \cdot 01125$.

$$a = \frac{2A}{t\{2 + (t-1)r\}} = \frac{3225 \cdot 25 \times 2}{28\{2 + (28-1) \times \cdot 01125\}} = \frac{6450 \cdot 50}{28 \times (2 + \cdot 30375)}$$

$= \dfrac{6450 \cdot 50}{28 \times 2 \cdot 30375} = \dfrac{6450 \cdot 50}{64 \cdot 505} = \$100 =$ quarterly payment, and hence annual annuity $= \$400$. *Ans.*

53. The application and interpretation of the remaining formulæ will be readily understood from the foregoing examples.

EXERCISE 163.

1. In what time will an annuity of \$1000 per annum, payable half-yearly, amount to \$8365, allowing simple interest, at the rate of 6 per cent. per annum? *Ans.* 14 payments, or 7 years.

NOTE.—In this question we use formula IV, r being equal to ·03 and $a = 500$.

2. If a rent of \$450 per annum, payable quarterly, be forborne for 11 years, to what does it amount, allowing 6 per cent. per annum, simple interest? *Ans.* \$6546·37½.

NOTE.—Take $a = \$112 \cdot 50$, $r = \cdot 015$ and $t = 44$.

3. At what rate per cent. per annum, simple interest, will an annuity of \$300, payable yearly, amount to \$1680 in 5 years? *Ans.* 6 per cent.

4. The rent of a farm is forborne for 8 years, and then amounts to \$2080. Now assuming the rent to be paid half-yearly, and simple interest at the rate of 8 per cent. per annum allowed, what was the rent of the farm? *Ans.* \$200.

ANNUITIES AT COMPOUND INTEREST.

54. Let A, a, r, $t =$ same quantities as in last articles, and also let $v =$ present value of the annuity.

Then, as before, the last payment of a forborne annuity being paid when due, $= a$; last payment but one, $= a +$ interest of a for one payment $= a + ar = a(1 + r)$; so also last payment but two, $= a(1 + r)^2$; last but three $= a(1 + r)^3$, &c., and first payment $= a(1 + r)^{t-1}$.

Hence A, the amount of the annuity $= a + a(1 + r) + a(1 + r)^2 + a(1+r)^3 +$ &c. $+ a(1+r)^{t-1}$, which is a geometrical series and is equal (Art. 16)

$$A = \frac{a\{(1+r)^t-1\}}{r}\ \text{(I)}$$

$$a = \frac{Ar}{(1+r)^t-1}\ \text{(II)}$$

$$r = \sqrt[t]{\frac{Ar+a}{a}} - 1\ \text{(III)}$$

$$t = \frac{log.\,(Ar+a) - log.\,a}{log.\,(1+r)}\ \text{(IV)}$$

$$v = \frac{a}{r}\left\{1 - \frac{1}{(1+r)^t}\right\}\ \text{(V)}$$

$$a = \frac{vr\,(1+r)^t}{(1+r)^t-1}\ \text{(VI)}$$

$$t = \frac{log.\,a - log.\,(a-vr)}{log.\,(1+r)}\ \text{(VII)}$$

$$v = \frac{a}{r}\left\{\frac{1}{(1+r)^s} - \frac{1}{(1+r)^{s+t}}\right\}\text{(VIII)}$$

$$v = \frac{a}{r}\ \text{(IX)}$$

$$a = vr\ \text{(X)}$$

$$r = \frac{a}{v}\ \text{(XI)}$$

$$v = \frac{a}{r(1+r)^s}\ \text{(XII)}$$

to $\frac{a\{(1+r)^t-1\}}{r}$, which is formula I of margin.

Formulas II, III, and IV are obtained from formula I by transposition, &c.

Since the present value of an annuity at compound interest is that principal which put out at compound interest for the given time, would produce the amount of the annuity we have from Art. 35, formula I,

$$v, (1 + r)^t = A = \frac{a\{(1 + r)^t - 1\}}{r}$$

whence by dividing by $(1+r)^t$, we get formula V in the margin.

Formulas VI and VII are derived from V.

To find the present value of an annuity which is to commence after t years and then continue for s years, we have from formula V, v for $s+t$ years, $=$ $\frac{a}{r}\left\{\frac{(1 + r)^{s+t} - 1}{(1+r)^{s+t}}\right\}$ and for t years alone, $v = \frac{a}{r}\left\{\frac{(1+r)^t-1)}{(1+r)^t}\right\}$

Therefore for t years to commence after s years. $v =$

$$\frac{a}{r}\left\{\frac{(1 + r)^{s+t} - 1}{(1+r)^{s+t}} - \frac{(1 + r)^t - 1}{(1+r)^t}\right\}$$

$$\text{or } v = \frac{a}{r}\left\{\frac{1}{(1 + r)^t} - \frac{1}{(1 + r)^{s+t}}\right\}$$

which is formula VIII in the margin.

When an annuity lasts for ever, as in the case of landed property, $(1+r)^t$ in formula V becomes infinitely great, and therefore $\frac{1}{(1+r)^t} = \frac{1}{\infty} = 0$, and the formula for finding the present value of a perpetuity is reduced to the form given in IX.

Formulas X and XI are derived from IX.

The present value of a freehold estate to a person to whom it will revert after s years and then continue for ever, is found from formula VIII and is represented by formula XII in the margin.

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55. To facilitate the calculation of annuities the following tables are given, the first showing the amount of an annuity of \$1 at compound interest, and the second, the present value of an annuity of \$1 at compound interest.

TABLE OF THE AMOUNTS OF AN ANNUITY OF $1 OR £1.

No. of Payments.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1......	1·00000	1·00000	1·00000	1·00000
2......	2·03000	2·04000	2·05000	2·06000
3......	3·09090	3·12160	3·15250	3·18360
4......	4·18363	4·24646	4·31012	4·37462
5......	5·30913	5·41632	5·52563	5·63706
6......	6·46841	6·63297	6·80191	6·97532
7......	7·66246	7·89829	8·14201	8·39384
8......	8·89234	9·21423	9·54911	9·89747
9......	10·15911	10·58279	11·02656	11·49131
10......	11·46388	12·00611	12·57789	13·18079
11......	12·80779	13·48635	14·20679	14·97164
12......	14·19203	15·02580	15·91713	16·86994
13......	15·61779	16·62684	17·71298	18·88214
14......	17·08632	18·29191	19·59863	21·01506
15......	18·59891	20·02359	21·57856	23·27598
16......	20·15688	21·82453	23·65749	25·67253
17......	21·76159	23·69751	25·84037	28·21288
18......	23·41443	25·64541	28·13238	30·90565
19......	25·11687	27·67123	30·53900	33·75999
20......	26·87037	29·77808	33·06595	36·78559
21......	28·67648	31·96920	35·71925	39·99273
22......	30·53678	34·24797	38·50521	43·39229
23......	32·45288	36·61789	41·43047	46·99583
24......	34·42647	39·08260	44·50200	50·81558
25......	36·45926	41·64591	47·72710	54·86451
26......	38·55304	44·31174	51·11345	59·15639
27......	40·70963	47·08431	54·66931	63·70576
28......	42·93092	49·96758	58·40258	68·52811
29......	45·21885	52·96629	62·32271	73·63980
30......	47·57541	56·08494	66·43885	79·05819
31......	50·00268	59·32833	70·76079	84·80168
32......	52·50276	62·70147	75·29829	90·88978
33......	55·07784	66·20953	80·06377	97·34316
34......	57·73018	69·85791	85·06696	104·18375
35......	60·46208	73·65222	90·32031	111·43478
36......	63·27594	77·59831	95·83628	119·12087
37......	66·17422	81·70225	101·62814	127·26812
38......	69·15945	85·97034	107·70954	135·90420
39......	72·23423	90·40915	114·09502	145·05846
40......	75·40126	95·02551	120·79977	154·76196
41......	78·66330	99·82654	127·83976	165·04768
42......	82·02320	104·81960	135·23175	175·95054
43......	85·48389	110·01238	142·99334	187·50758
44......	89·04841	115·41288	151·14300	199·75803
45......	92·71986	121·02939	159·70015	212·74351
46......	96·50416	126·87957	168·68516	226·50812
47......	100·39650	132·94539	178·11924	241·09861
48......	104·40839	139·26321	188·02539	256·56453
49......	108·54065	145·83373	198·42666	272·95840
50......	112·79687	152·66708	209·34799	290·33590

TABLE OF PRESENT VALUES OF AN ANNUITY OF $1 OR £1.

No. of Payments.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1......	0·97097	0·96154	0·95238	0·94340
2......	1·91347	1·88619	1·86941	1·83339
3......	2·82861	2·77519	2·87519	2·67301
4......	3·71710	3·62999	3·54595	3·46510
5......	4·57971	4·45182	4 32948	4·21236
6......	5·41719	5·24214	5·07569	4·91732
7......	6·23028	6·00205	5·78637	5·58238
8......	7 01969	6·73274	6·46321	6·20979
9......	7·78611	7·43533	7·10782	6·80169
10......	8·53920	8·11089	7·72173	7·36009
11......	9·25262	8·76058	8·30641	7·88687
12......	9·95400	9·38507	8 86325	8·38384
13......	10·63496	9·98565	9·39357	8·85268
14......	11·29607	10·56312	9·89864	9·29498
15......	11 93794	11·11849	10·37965	9·71225
16......	12·56110	11·65239	10·83777	10·10589
17......	13·16612	12·16567	11·27406	10·47726
18......	13·75351	12·65940	11·68958	10 82760
19......	14·32380	13 13394	12·08532	11·15811
20......	14·87748	13·59032	12·46221	11·46992
21......	15·41502	14·02916	12·82115	11·76407
22......	15·93692	14·45111	13·16300	12·04158
23......	16·44361	14·85643	13·48857	12·30338
24......	16·93554	15·24696	13·79864	12·55036
25......	17·41315	15·62208	14·09394	12·78335
26......	17·87684	15·98277	14·37518	13·09316
27......	18·32703	16·32958	14·64303	13·21053
28......	18·76411	16·66306	14·89812	13·40616
29......	19·18846	16·98371	15·14107	13·59072
30......	19·60044	17·29203	15·37245	13·76483
31......	20·00043	17·58849	15·59281	13·92908
32......	20 38877	17·87355	15·80267	14·08404
33......	20·76579	18·14764	16·00255	14·23023
34......	21·13184	18 41119	16·19290	14·36814
35......	21·48722	18·66461	16·37419	14·49824
36......	21·83225	18·90828	16·64685	14·62099
37......	22·16724	19·14258	16·71128	14·73678
38......	22·49246	19 36786	16·86789	14·84602
39......	22·80822	19·58448	17·01704	14·94907
40......	23·11477	19·79277	17·15908	15·94630
41	23·41240	19·99305	17·29436	15·13801
42......	23·70136	20·18562	17·42320	15·22454
43......	23 98190	20·37079	17·54591	15·30617
44......	24·25428	20·54844	17·66277	15·38318
45......	24·51871	20·72004	17·77407	15·45583
46......	24·77545	20·88465	17·88006	15·52437
47......	25·02471	21·04293	17·98101	15·58903
48......	25·26677	21·19513	18 07714	15·65002
49......	25·59166	21·50166	18·16872	15·70757
50......	25·72977	21·72977	18·25592	15·76186

APPLICATIONS.

56. To find the amount of an annuity forborne for any number of years at compound interest:

RULE.

$$A=\frac{a\{(1+r)^t-1\}}{r}\quad(\text{I.})$$

INTERPRETATION.—*From the amount raised to the power indicated by the number of payments subtract 1 and multiply the remainder by the annuity. Lastly: divide the sum thus obtained by the rate per unit and the quotient will be the required amount.*

BY THE TABLE.—*Find from the table the amount of $1 for the given number of payments and at the given rate; multiply it by the given annuity and the quotient will be the amount.*

EXAMPLE.—If a yearly rent of $400 be forborne for 23 years, to what sum will it amount at 5 per cent. compound interest?

OPERATION.

Here $a=400$, $t=23$, $r={\cdot}05$.

Then $A=\dfrac{a\{(1+r)^t-1\}}{r}=\dfrac{400\{(1{\cdot}05)^{23}-1\}}{{\cdot}05}=\dfrac{400\times 2{\cdot}071475}{{\cdot}05}=\dfrac{828{\cdot}590}{{\cdot}05}$
$=\$16571{\cdot}80$. *Ans.*

BY THE TABLE.—Amount of $1 at the given rate and time = $41·43047.
Then $41·43047 × 400 = $16572·188.

NOTE.—These two methods give results slightly different. This arises from the fact that the table shows only an approximation to the correct amount of the annuity for $1; all the figures except the first five of its decimal being rejected.

57. To find the present value of an annuity at compound interest:—

RULE.

$$v=\frac{a}{r}\left\{1-\frac{1}{(1+r)^t}\right\}\quad(\text{V.})$$

INTERPRETATION.—*Divide one by that power of the amount of $1 which is indicated by the number of payments and subtract the result from 1.*

Multiply the remainder by the quotient arising from the division of the given annuity by the rate per unit and the result will be the required present value.

BY THE TABLE.—*Find the present value of an annuity of $1 for the given number of payments and at the given rate, and multiply this by the given annuity.*

EXAMPLE.—What is the present value of an annuity of $40, to continue 5 years, allowing 5 per cent. compound interest?

OPERATION.

Here $a = 40$, $t = 5$, and $r = ·05$.

Then $v = \frac{a}{r}\left\{1 - \frac{1}{(1+r)^t}\right\} = \frac{40}{·05} \times \left\{1 - \frac{1}{(1·05)^5}\right\} = \frac{4000}{5} \times 1 - ·7835)$ $= 800 \times ·2165 = \$178·20$. *Ans.*

OR BY THE TABLE.—Present value of an annuity of $1 for given rate and time = $4·82948 and $4·82948 × 40 = $178·179. *Ans.*

58. To find the present worth of a perpetuity :—

RULE.

$$V = \frac{a}{r}. \quad \text{(IX.)}$$

INTERPRETATION.—*Divide the annuity by the rate per unit and the quotient will be the value of the perpetuity.*

EXAMPLE.—What is the present value of a freehold estate of $75—allowing the purchaser 6 per cent. compound interest for his money?

OPERATION.

Here $a = 75$, and $r = ·06$

Then $V = \frac{a}{r} = \frac{75}{·06} = \frac{7500}{6} = \$1250$. *Ans.*

59. To find the present worth of a perpetuity in reversion :—

RULE.

$$V = \frac{a}{r(1+r)^s} \quad \text{(XII.)}$$

INTERPRETATION.—*Find that power of the amount of $1 for one ayment that is indicated by the number of payments that have to lapse before the annuity reverts, multiply this by the rate per unit nd divide the given annuity by the product—the result will be the resent value.*

EXAMPLE.—What is the present value of the reversion of a peretuity of $79·20 per annum, to commence 7 years hence—allowing he buyer 4½ per cent. for his money?

OPERATION.

Here $a = 79·20$, $s = 7$, and $r = ·045$.

Then $V = \frac{a}{r(1+r)^s} = \frac{79·20}{·045 \times (1+·045)^7} = \frac{79·20}{·045 \times 1·360862} = \frac{79·20}{·06123879} =$ 298·297. *Ans.*

60. With due attention to the foregoing interpretations and examples, the pupil will not experience any difficulty in applying the remaining formulæ.

EXERCISE 164.

1. What is the annual rental of a freehold estate, purchased for $3000 when the rate of interest is at 4 per cent.? *Ans.* $120.
2. If a perpetuity of $563 can be purchased for $11260 ready money, what is the rate of interest allowed? *Ans.* 5 per cent.
3. A freehold estate producing $75 per annum is mortgaged for the period of 14 years; what is its present value, reckoning compound interest at 5 per cent. per annum? *Ans.* $757·608.
4. Required the present value of a deferred annuity of $90, to be entered upon at the expiration of 12 years, and then to be continued for 7 years at 4 per cent. compound interest. *Ans.* $337·39.
5. What is the present value of an estate whose rental is $1500, allowing 5 per cent. compound interest? *Ans.* $30000, or 20 years' purchase.
6. For how many years may an annuity of £22 be purchased for £308 12s. 10d., allowing compound interest at 4 per cent.? *Ans.* 21 years.
7. What is the present value of an annuity of $154 for 19 years at 5 per cent. compound interest? *Ans.* $1861·13.
8. What annuity, accumulating at 3¼ per cent. compound interest, will amount to £600 in 40 years? *Ans.* £6 13s. 11d.
9. In how many years will an annuity of $8 per annum amount to $187·315625 at 3 per cent. compound interest? *Ans.* 18 years.
10. What will an annuity of $74 amount to in 30 years at 4 per cent. compound interest? *Ans.* $4150·28.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the numbered articles of the section.*

1. When are quantities said to be in arithmetical progression? (1)
2. What are the extremes? What the means? (2)
3. What five quantities are to be considered in arithmetical progression? (3)
4. How are these related to each other? (3)
5. How many cases arise from these combinations? (3)

6. Deduce the fundamental formulæ for arithmetical progression. (4-7)
7. When are quantities said to be in geometrical progression? (15)
8. What five quantities are to be considered in geometrical progression? (16)
9. How are these related, and how many cases arise from their combinations? (16)
10. Deduce the fundamental formulæ for geometrical progression. (17-19)
11. What rule do you use when finding the sum of any infinite series when the ratio is less than 1? (25)
12. Prove this rule. (25)
13. How do we insert any number of arithmetical means between two given extremes? (26)
14. How do we insert any number of geometrical means between two extremes? (26)
15. What is position? (27)
16. Into what rules is position divided? (28)
17. When is a single position used? (29)
18. What class of questions require the use of double position? (30)
19. Give and prove the common rule for single position. (32)
20. Give and prove the common rule for double position. (34)
21. Deduce algebraically a complete set of rules for compound interest. (35)
22. What is an annuity? (40)
23. When is an annuity said to be in possession? (41)
24. What is a deferred annuity or an annuity in reversion? (42)
25. What is a contingent annuity? (44)
26. What is a perpetuity? (45)
27. When is an annuity said to be in arrears? (46)
28. What is the amount of an annuity? (47)
29. What is the present worth of an annuity? (48)
30. Deduce a set of rules for computing annuities at simple interest.
31. Illustrate the absurdity and injustice of computing the present value of annuities at simple interest. (50)
32. Deduce a set of rules for annuities at compound interest. (54)

EXERCISE 165.

EXAMINATION PROBLEMS.

FIRST SERIES.

1. Write down as one number seven trillions and ninety millions, and nineteen and four million two hundred thousand and six hundredths of trillionths.
2. Deduct 19 per cent. from $7580 and divide the remainder among A, B, C, and D, so that A may have $111·11 more than B; B $90·90 more than C, and D one third as much as A, B and C together.
3. A and B can perform a piece of work in 8 days, when the days are 12 hours long; A, by himself, can do it in 12 days, of 16 hours each. In how many days of 14 hours long will B do it?
4. Reduce £179 14s. 8½d. to dollars and cents, and divide the result by ·00000048.
5. What is the l. c. m. of 44, 18, 80, 77, 56 and 27?

6. In what time will any sum of money amount to 20 times itself at $5\frac{1}{2}$ per cent. simple interest?
7. Divide 7342163 *octenary* by 61351 *nonary*, and give the answer in the duodenary scale true to two places to the right of the separating point.
8. Multiply 43 lbs. 8 oz. 17 dwt. 11 grs. by $783\frac{1}{2}$.
9. Find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, *ad infinitum.*
10. Divide $\frac{1}{2}$ of $\frac{2}{3}$ of 192 by $\dfrac{\;\dfrac{2\frac{1}{2}}{3}\;}{\dfrac{4\frac{1}{2}}{\dfrac{2}{\frac{4}{5}}}}$
11. Extract the 17th root of 129140163.
12. There is a number consisting of two places of figures, which is equal to four times the sum of its digits, and if 18 be added to it, its digits will be inverted. What is the number?

SECOND SERIES.

13. Divide \$897·43 among A, B, and C, so that B may have \$93·40 less than A, and \$69·18 more than C.
14. If 7 lbs. of wheat contain as much nutritive matter as 9 lbs. of rye, and 5 lbs. of rye as much as 8 lbs. of oats, and 13 lbs. of oats as much as 21 lbs. of buckwheat, and 27 lbs. of buckwheat as much as 20 lbs. of barley, and 24 lbs. of barley as much as 26 lbs. of peas, and 11 lbs. of peas as much as 35 lbs. of potatoes; how many pounds of potatoes contain as much nourishment as 16 lbs. of wheat?
15. Reduce $\frac{2}{3}$ of $4\frac{1}{2}$ of $7\frac{4}{5}$ of $\dfrac{9}{19\frac{1}{3}}$ of $\frac{5}{9}$ of 3 oz. 4 drs. 2 scr. 5 grains to the decimal of $\frac{6}{11}$ of $\cdot\dot{6}\dot{3}$ of $2\frac{37}{42}$ of $\frac{3}{13}$ of $6\frac{1}{2}$ times 7 lbs. 3 oz., Apothecaries' Weight.
16. From $623\cdot42\dot{7}9\dot{3}$ take $93\cdot42671\dot{9}\dot{2}$; mark distinctly the resulting repetend.
17. If I own a vessel valued at \$7493 and wish to insure it at a premium of $4\frac{3}{5}$ per cent. so as to recover, in case of the destruction of the vessel, both the premium paid and the value of the vessel, for what sum must I insure?
18. If 18 men in 20 weeks of 5 working days each, working 11 hours a day, dig 11 cellars, each 20 feet long, 16 feet wide

and 5 feet deep; how many men will be required to dig 24 cellars, each 22 feet square and 4 feet deep, in 36 weeks of 6 days each, working 9 hours per day?

19. A certain number is divided by 9 and the quotient multiplied by 17; the product is then divided by 300 and 33 is added to the quotient; the result is next divided by 3, and from this quotient 31 is subtracted, and the reulting difference divided by 12½. Now ½ of ⅗ of $\frac{4}{7}$ of this last quotient is $2\frac{2}{35}$. Required the original number.
20. What is the l. c. m. of 480, 768, 348, and 1176?
21. What is the G. C. M. of 17598, 46090, and 171847?
22. In a certain adventure A put in $12000 for 4 months, then adding $8000, he continued the whole two months longer; B put in $25000, and after three months took out $10000, and continued the rest for 3 months longer; C put in $35000 for 2 months, then withdrawing $\frac{2}{7}$ of his stock, continued the remainder for 4 months longer; they gained $15000; what was the share of each?
23. Three merchants traffic in company, and their stock is £400; the money of A continued in trade 5 months, that of B 6 months, and that of C 9 months; and they gained £375, which they divide equally. What stock did each put in?
24. A fountain has 4 pipes, A, B, C, and D, and under it stands a cistern, which can be filled by A in 6, by B in 8, by C in 10, and by D in 12 hours; the cistern has 4 pipes, E, F, G, and H; and can be emptied by E in 6, by F in 5, by G in 4, and by H in 3 hours. Suppose the cistern is full of water, and that 8 pipes are all open, in what time will it be emptied?

THIRD SERIES.

25. Express 74938 and 17498679 in Roman Numerals.
26. 2310 loaves of bread are divided among charitable institutions in the following manner: as often as the first receives 4 the second receives 3, and as often as the first receives 6 the third gets 7; how many will each have?
27. How much sugar at 4, 5, and 9 cents a pound, must be mixed with 72 pounds at 12 cents a pound, so that the mixture may be worth 8 cents a pound?
28. What principal put out at simple interest will amount to $4444·44 in 4 years 4 months 4 days at 4·44 per cent.?
29. For what sum must a ship valued at $23470 be insured so as, in case of its destruction, to recover both the value of the vessel and the premium of 2½ per cent.?

30. What principal will amount to $7498·47 in 8 years, allowing simple interest at 7 per cent.?

31. I send to my agent in Manchester $17460 and instruct him to deduct his commission at $3\frac{1}{3}$ per cent., and invest the balance in broadcloths at $2·95 per yard. When I receive the goods I have to pay in addition $1347·90 for carriage, $479·40 for insurance, $169·83 for storage, wharfage, and harbour dues, and an *ad valorem* duty at $2\frac{1}{2}$ per cent. on the invoice of goods. Required how many yards of cloth my agent ships to me and what I gain or lose per cent. on the whole transaction if I sell the goods for $25000.

32. Transpose 134234 *quinary* into the *ternary*, *octenary*, and *duodenary* scales, and prove the results by reducing all four numbers to the *denary* scale.

33. What is the difference between $\frac{4}{7}$ of $4\frac{1}{2}$ of $\frac{9\frac{2}{3}}{\frac{13}{20}}$ of $\frac{1}{15}$ of $\frac{7}{9}$ of £43 18s. $11\frac{1}{2}$d., and $3\frac{3}{8}$ of $\frac{1}{17\frac{1}{2}}$ of ·56 of 1·75 of $6\frac{1}{2}$ times $97·18?

34. Given the logarithm of 2=0·301030
3=0·477121
13=1·113943
Find the logarithms of $\frac{1}{13}$, 19·5, 1125, 28·16, 65000, ·0005, 152·1, and 8·112.

35. Extract the cube root of 871*tet*·72 *duodenary* true to two places to the right of the separating point.

36. A person passed $\frac{1}{6}$ of his age in childhood, $\frac{1}{12}$ of it in youth, $\frac{1}{7}$ of it + 5 years in matrimony; he had then a son whom he survived 4 years, and who reached only $\frac{1}{2}$ the age of his father. At what age did this person die?

FOURTH SERIES.

37. Divide 63 miles 3 fur. 7 per. 3 yds. 2 ft. 7 in. by 7 fur. 23 per. $3\frac{3}{4}$ yds.

38. Divide 6·3 by ·000000274.

39. If $\frac{7}{8}$ yards of cloth cost $$\frac{4}{7}$, how much will $6\frac{3}{11}$ yards cost?

40. Find the interest on $4237·71 at $6\frac{1}{2}$ per cent. for 1·67 years.

41. In what time will $674·30 amount to $1000 at $8\frac{1}{2}$ per cent.?

42. What are the amount and compound interest of $818·71 for 7 years at 4 per cent. half-yearly?

43. A owes B $4300 to be paid as follows, viz.: $300 down, $700 at the end of 4 months, $750 at the end of 7 months, $850 at the end of 9 months, $400 at the end of 13 months, and the balance at the end of 19 months. Required the equated time for the whole debt.

44. Deduct 23 per cent. from $4200 and divide the remainder between A, B, C, D, and E, so that A may have $17·10 more than B, C $19·28 less than B, D $42·11 less than C, and E half as much as A, B, C, and D together.

45. What principal put out at simple interest at 16 per cent. will amount to $3786·80 in 11 years?

46. Find the value of

$$\frac{\left\{(3\tfrac{3}{7}-2\tfrac{7}{10})\times\cdot\dot{4}\dot{6}\div\tfrac{3}{8}\text{ of }\cdot\dot{1}4285\dot{7}\right\}+8\tfrac{1}{2}\text{ times }\left(\tfrac{1}{2}+\tfrac{1}{7}+\tfrac{1}{5}-\tfrac{337}{2310}\right)}{\left\{(\cdot7\dot{8}\times\cdot123\dot{4}\dot{5}+\tfrac{678}{750})+\tfrac{3}{7}+9\tfrac{3}{8}+17\tfrac{4}{11}\right\}\div27.49220\dot{7}\dot{7}}$$

47. Add together 312312302 and 2312132 *quaternary;* multiply the sum by twenty-three thousand and eleven times 4234 *quinary;* from the product subtract 555 + 444 + 333 + 222 + 111 *senary;* divide the remainder by 6542 *septenary,* and give the answer in the *octenary* scale.

48. What is the square of ·1 and also of $\cdot\dot{1}$?

FIFTH SERIES.

49. Read the following numbers:

1000300500600·00070080009.
7600290034007·000000067400209.

50. Find the l. c. m. of 2, 9, 16, 27, 48, and 81.

51. In what time will any sum of money amount to 7 times itself at 6 per cent. per annum compound interest?

52. How often will a coach wheel turn in going from Toronto to Brampton, a distance of 20 miles; the wheel being 14 ft. 10 in. in circumference?

53. How many divisors has the number 1749600?

54. Divide $\frac{3}{8}$ of $\dfrac{96}{\frac{5}{6}}$ by $\dfrac{\frac{1}{2}\text{ of }7}{\frac{3\frac{1}{4}}{2}}$

55. A can do a piece of work in 12 days, and A and B together can do it in 5 days; in what time can B alone do it?

56. What principal will amount to $8899·77 in 11 years at 6 per cent. half yearly, compound interest?

57. Divide the number 10 into three such parts, that if the first be multiplied by 2, the second by 3, and the third by 4, the three products will be equal.

58. There are three fishermen, A, B, and C, who have each caught a certain number of fish; when A's fish and B's are put together, they make 110; when B's and C's are put together they make 130; and when A's and C's are put together they make 120. If the fish be divided equally among them, what will be each man's share; and how many fish did each of them catch?

59. What is the forty-seventh term and also the sum of the first 98 terms of the series 7, 11, 15, 19, &c.?

60. In what time will any sum of money amount to 21 times itself at 7 per cent. compound interest?

SIXTH SERIES.

61. Divide $3700 among three persons, A, B, and C, so that B may have $387 less than A and $196·87 more than C.

62. What are all the divisors of 5716?

63. What is the value of

$$\frac{\left\{(17\tfrac{7}{12}-10\tfrac{59}{60})-(\cdot4+\tfrac{1}{5}+\cdot9-\tfrac{1}{4})\right\}+(\cdot8\dot{3}78\div\tfrac{1}{3}\text{ of }31)}{\cdot63\dot{2}263\dot{2}\times\tfrac{1}{3}\text{ of }\cdot9\tfrac{1}{4}+(\tfrac{1}{5}\text{ of }4\tfrac{1}{6}\text{ of }\tfrac{1}{11}\text{ of }85\tfrac{16}{37}+101)}$$

64. Divide $7200 among 3 men, 4 women, and 17 children giving each man twice as much as a woman, and each woman three times as much as a child. What is the share of each?

65. How many divisors has the number 25400?

66. What is the difference between $\frac{2}{3}$ of $4\frac{1}{2}$ of $\frac{9\frac{3}{7}}{1\frac{1}{4}}$ of $\frac{1}{6}$ of £[illegible] 16s. 11[illegible]d. and $\frac{3}{11}$ of $4\frac{2}{5}$ of $\frac{19\frac{1}{3}}{\frac{3\frac{1}{4}}{1\frac{3}{9}}}$ of $\frac{85}{117}$ of $\frac{11}{23}$ of ·85 of $\frac{1}{42\frac{1}{2}}$ of $1783?

67. Compare together the ratios 7 : 13, 9 : 16, 8 : 15 and 10 : 19 and point out which is the greatest, which the least, and what the ratio compounded of these given ratios.

68. Divide $67\cdot\dot{4}3\dot{2}$ by $7\cdot9\dot{0}3\dot{6}$.

69. Reduce 9 per. 9 yds. 7 ft. 120 in. to the decimal of $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{3}{7}$ of 35 acres 2 roods.

70. Add together $17\cdot03\dot{4}\dot{2}$, $27\cdot063\dot{5}\dot{7}$, $98\cdot123\dot{4}5\dot{6}$, $829\cdot6\dot{4}2\dot{3}$, $986\cdot123\dot{4}29\dot{8}$, $9\cdot\dot{8}7634\dot{2}$, and $813\cdot98642\dot{3}456\dot{7}$.

71. In the ruins of Persepolis there are two columns left standing upright. The one is 64 feet above the plain and the other 50. In a straight line between these stands a small statue, the head of which is 97 feet from the top of the higher column and 86 feet from the top of the lower, the base of which is 76 feet from the base of the statue. Required the distance between the tops of the columns.

72. In a mixture of spirits and water, $\frac{1}{2}$ of the whole plus 25 gallons was spirits, but $\frac{1}{3}$ of the whole minus 5 gallons was water. How many gallons were there of each?

SEVENTH SERIES.

73. Extract the square root of 401241·3424 in the *quinary* scale.

74. A father being asked by his son how old he was, replied, your age is now ⅓ of mine; but 4 years ago it was only ¼ of what mine is now: what is the age of each?

75. Divide $\cdot 7\dot{2}34\dot{7}$ by $\cdot 00\dot{3}\dot{2}$.

76. Extract the 11th root of 97294764·372.

77. Find two numbers, the difference of which is 30, and the relation between them as 7¼ is to 3½.

78. What is the l. c. m. of 35, 16, 18, 3 and 40?

79. Sum the series 1+7+13+19+&c terms.

80. What is the ratio compounded of : 56, 35 : 121, 11$\frac{3}{7}$: 29, 8 : 4¾ and 4⅐ : 3?

81. Find two numbers whose sum an oduct are equal, neither of them being 2.

NOTE.—In this question take any number for the first of the two, as for example 7. Then 7+some other number=7 × that other number.
Assume for this second number any other, as 3.
Then 7+3=10=7 × 3, gives an error of−11.
ne some other for the second, as 5.
Then 7 + 5 =12=7 × 5, gives an error of−23.
Then 23 × 3=69
11 × 5=55 Whence second number=$\frac{14}{12}$=1⅙.

82. Find the value of

$$\frac{\left(\left\{\left(9\tfrac{1}{5}+4\tfrac{11}{12}+3\tfrac{1}{7}-16\tfrac{31}{35}\right)\times\cdot\dot{5}\dot{4}\right\}+1\tfrac{4}{7}\right)\times 35 \text{ times } \cdot\dot{1}4285\dot{7}.}{\left\{\cdot 9\dot{7}\times\cdot 24\dot{3}7\dot{8}\times\left(1\tfrac{1}{44}\times 4\tfrac{46}{451}\right)\right\}\times\left(4\tfrac{3}{11}-2\tfrac{4}{17}\right).}$$

83. The hour and minute hands of a watch are together at 12; when will they be together again?

84. Given the logarithm of 2=0·301030
logarithm of 7=0·845098
logarithm of 11=1·041393

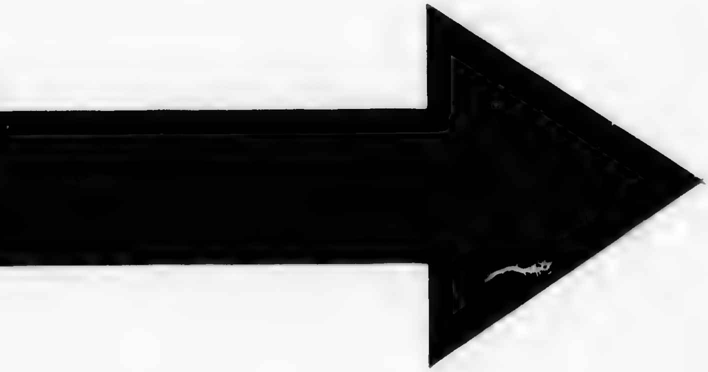
Find the logarithms of 3850000, 3181·$\dot{8}\dot{1}$, ·0000154, $\frac{1}{77}$, $1\cdot\dot{5}7142\dot{8}$ and 93·17

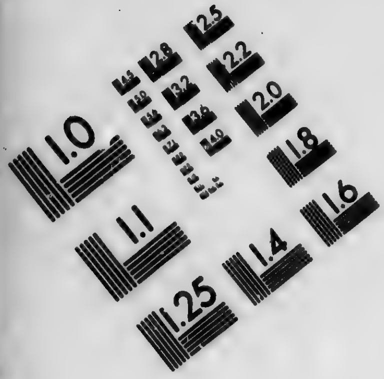
EIGHTH SERIES.

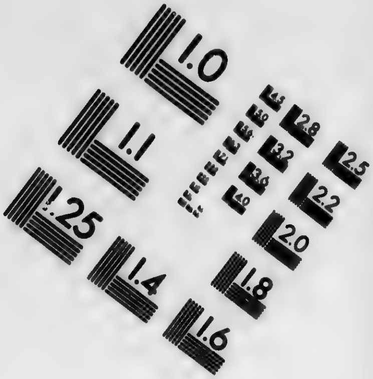
85. Find the difference between the simple and compound interest of $700 in 3 years at 4½ per cent. per annum.

86. X, Y, and Z, form a company. X's stock is in trade 3 months, and he claims $\frac{1}{12}$ of the gain; Y's stock is 9 months in trade; and Z advanced $3024 for 4 months, and claims half the profit. How much did X and Y contribute?

[redacted]







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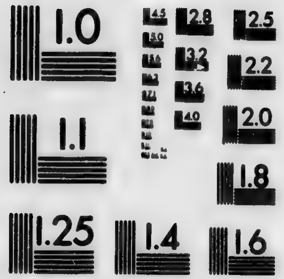

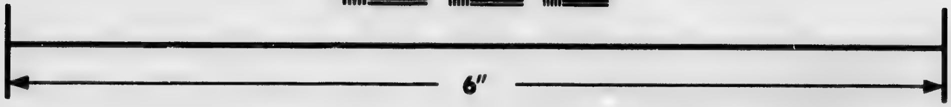

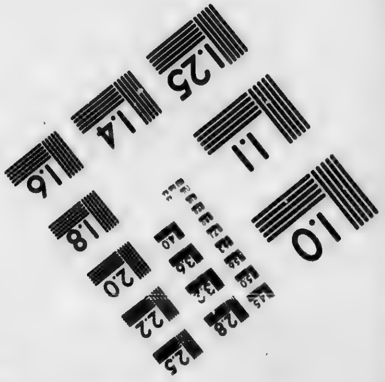

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87. There is a fraction which multiplied by the cube of 1⅓ and divided by the square root of $1\frac{7}{9}$, produces ⅔; find it.
88. Find the cube root of 80677568161.
89. How much sugar, at 4d., 6d., and 8d. per lb. must there be in 112 lbs. of a mixture worth 7d. per lb.
90. Find three such numbers as that the first and ½ the sum of the other two, the second and ⅓ the sum of the other two, the third and ¼ the sum of the other two, will make 34.

NOTE.—Assume 40 as the *sum* of the three numbers.
Then 1st + 2nd + 3rd = 40 and 1st + ½ (2nd + 3rd) = 34 ∴ ½ (2nd + 3rd) = 6 and 2nd + 3rd = 12.
2nd + ⅓ (1st + 3rd) = 34 ∴ ⅔ (1st + 3rd) = 6 and 1st + 3rd = 9.
3rd + ¼ (1st + 2nd) = 34 ∴ ¾ (1st + 2nd) = 6 and 1st + 2nd = 8.
Then adding these together, twice (1st + 2nd + 3rd) = 29 ∴ 1st + 2nd + 3rd = 14½ = sum.
But should equal 40—therefore error = −25½.
Similarly assume some other number and apply the rule, and the true sum 58 will be found, from which the numbers may be easily obtained.

91. Insert 4 arithmetical means between 1 and 40.
92. The sum of all the terms of a geometrical progression is 1860040, the last term is 1240029, and the ratio is 3. What is the first term?
93. If 6 apples and 7 pears cost 33 pence, and 10 apples and 8 pears 44 pence, what is the price of one apple and one pear?
94. Multiply ⅓ of ⅔ of ⅝ of $\frac{28\frac{1}{3}}{6}$ by ⅔ of ¾ of ⅞.
95. From a sum of money, $50 more than the half of it is first taken away; from the remainder, $80 more than its fifth part; and again from the second remainder, $20 more than its fourth part. At last there remained only $10. What was the original sum?
96. A gentleman hires a servant, and promises him, for the first year, only $60 in wages, but for each following year $4 more than the preceding. How much will the servant receive for the 17th year of his engagement, and how much for all 17 years together?

NINTH SERIES.

97. Write down as one number eleven trillions and eleven, and eleven tenths of billionths.
98. Reduce £749 16s. 5¾d. sterling to dollars and cents.
99. What are the prime factors of 177408?
100. At what rate per cent. per annum will $704 amount to $11111·11 in 11 years at compound interest?

101. How many scholars are there in a school to which if 9 be added the number will be augmented by one-thirteenth?

102. Three different kinds of wine were mixed together in such a way that for every 3 gallons of one kind there were 4 of another, and 7 of a third: what quantity of each kind was there in a mixture of 292 gallons?

103. Divide £500 among four persons, so that when A has £$\frac{1}{2}$, B shall have £$\frac{1}{3}$, C $\frac{1}{4}$, and D $\frac{1}{5}$.

104. What is the present worth of an annuity of $100 to continue 23 years, at 6 per cent. compound interest?

105. Twenty-five workmen have agreed to labor 12 hours a day for 24 days, to pay an advance made to them of $900; but having each lost an hour per day, five of them engage to fulfil the agreement by working 12 days: how many hours per day must these labor?

106. A man has several sons, whose ages are in arithmetical progression; the age of the youngest is 5 years, the common difference of their ages is 6 years, and the sum of all their ages is 161. What is the age of the eldest?

107. If a man dig a small square cellar, which will measure 6 feet each way, in one day, how long will it take him to dig a similar one that shall measure 10 feet each way?

108. A servant agreed to live with his master for £8 a year, and a suit of clothes. But being turned out at the end of 7 months, he received only £2 13s. 4d. and the suit of clothes: what was its value?

TENTH SERIES.

109. What number is that of which $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{4}$ added together, will make 48?

110. If an ox, whose girth is 6 feet, weighs 600 lbs., what is the weight of an ox whose girth is 8 feet?

111. Four women own a ball of butter, 5 inches in diameter. It is agreed that each shall take her share separately from the surface of the ball. How many inches of its diameter shall each take?

112. Divide 71214·48 by 12·342 in the *nonary* scale and extract the square root of the quotient true to three places to the right of the separating point.

113. Five merchants were in partnership for four years; the first put in $60, then, 5 months after, $800, and at length $1500, four months before the end of the partnership; the second put in at first $600, and six months after $1800; the third put in $400, and every six months after he added $500; the fourth did not contribute till 8 months after the commencement of the partnership; he then put in \$900, and repeated this sum every six months; the fifth put in no capital, but kept the accounts, for which the others agreed to pay him \$1·25 a day. What is each one's share of the gain, which was \$20000?

114. In what time will any sum of money amount to 16 times itself at five per cent. per annum. 1st. at simple interest? 2nd. at compound interest?

115. Three persons purchased a house for \$9202; the first gave a certain sum; the second three times as much; and the third one and a half times as much as the two others together: what did each pay?

116. A piece of land of 165 acres was cleared by two companies of workmen; the first numbered 25 men and the second 22; how many acres did each company clear, and what did the clearing cost per acre, knowing that the first company received \$86 more than the second?

117. The greater of two numbers is 15 and the sum of their squares is 346: what are the two numbers?

118. To what sum will \$1200 amount in 10 years at $9\frac{1}{2}$ per cent. simple interest?

119. If 496 men, in $6\frac{1}{2}$ days of 11 hours each, dig a trench of 7 degrees of hardness, 465 feet long, $3\frac{2}{3}$ wide, $2\frac{1}{3}$ deep, in how many days of 9 hours long will 24 men dig a trench of 4 degrees of hardness, $337\frac{1}{2}$ feet long, $5\frac{3}{5}$ wide, and $3\frac{1}{2}$ deep?

120. Four men, A, B, C, and D, took a prize of \$6213, which they are to divide in proportion to the following fractions: *if possible*, A, B, and C, are to have $\frac{47}{60}$; B, C, and D, $\frac{37}{60}$; A, C, and D, $\frac{7}{10}$; and A, B, and D, $\frac{3}{4}$ of the prize. What does each receive?

ELEVENTH SERIES.

121. Reduce $\cdot\dot{7}$, $\cdot\dot{8}\dot{3}$, $\cdot\dot{7}2\dot{7}$, $\cdot9\dot{1}32\dot{5}$ and $8\cdot67\dot{1}34\dot{7}$ to their equivalent vulgar fractions.

122. Reduce $713\frac{2}{3}\frac{9}{t}\frac{t}{1}$ *undenary*, and $12123\frac{11233}{100000}$ *quaternary* to equivalent expressions in the *denary* scale.

123. Add together $3\frac{3}{8}$ of $2\frac{1}{3}$ of $7\frac{1}{2}\frac{1}{0}$ of a £, $9\frac{3}{7}$ of $3\frac{5}{9}$ of a shilling, and $8\frac{1}{4}$ of $4\frac{1}{3}$ of a penny, and divide the sum by $\frac{11}{12}$ of $5\frac{5}{14}$ of $\frac{2}{3}$ of $8\frac{1}{2}$d.

124. If 24 pioneers, in $2\frac{1}{2}$ days of $12\frac{1}{2}$ hours long, can dig a trench 139 75 yds. long, $4\frac{1}{2}$ yds. wide, and $2\frac{1}{2}$ yds. deep, what length of trench will 90 pioneers dig in $4\frac{1}{3}$ days of $9\frac{2}{3}$ hours long, the trench being $4\frac{7}{5}$ yds. wide, and $8\frac{1}{3}$ yds. deep?

125. A person, by disposing of goods for \$182, loses at the rate of 9 per cent.: what ought they to have been sold for to realize a profit of 7 per cent.?

126. In what time will any sum of money amount to $11\frac{1}{2}$ times itself at 6 per cent. per annum?
1st At simple interest?
2nd At compound interest?

127. It is desired to cut off an acre of land from a field $15\frac{1}{2}$ perches in breadth; what length must be taken?

128. Express a degree ($69\frac{1}{12}$ miles) in metres, when 32 metres are equal to 35 yards?

129. Find 7 geometrical means between 3 and 19683.

130. Sum the infinite series $7 + 1\frac{3}{4} + \frac{7}{16}$, &c.

131. Four men bought a grindstone of 60 inches diameter. Now, how much of the diameter must be ground off by each man, one grinding his part first, then another, and so on, that each may have an equal share of the stone, no allowance being made for the axle?

132. Divide 100 guineas into an equal number of guineas, half-guineas, crowns, half-crowns, shillings, and sixpences, and reduce the remainder to a fraction of a pound.

TWELFTH SERIES.

133. The owner of $\frac{4}{17}$ of a ship sold $\frac{3}{11}$ of $\frac{4}{9}$ of his share for $\$12\frac{4}{33}$; what would $\dfrac{2\frac{1}{2}}{4\frac{1}{4}}$ of $\frac{3}{8}$ of the ship cost at the same rate?

134. At what rate per cent. per annum will \$700·90 amount to \$1679·40 in 5 years, compound interest being allowed?

135. A person paid a tax of 10 per cent. on his income; what must his income have been, when, after he had paid the tax, there was \$1250 remaining?

136. The sum of £3 13s. 6d. is to be divided among 21 men, 21 women, and 21 children, so that a woman may have as much as two children, and a man as much as a woman and a child: what will each man, woman, and child receive?

137. Distribute \$200 among A, B, C, and D, so that B may receive as much as A; C as much as A and B together, and D as much as A, B, and C together.

138. Find the difference between $\sqrt{\frac{2}{3}}$ and $\sqrt[3]{\frac{2}{3}}$.

139. Reduce $\frac{3473}{92807}$, $17\frac{5}{12} + \frac{4}{15} + 144\frac{11}{21}$, $2\frac{13}{35} - \frac{17}{25}$, $\frac{3}{4}$ of $\frac{5}{7} \times \frac{4}{15}$ of $\frac{15}{11}$ of $\frac{21}{23}$, and $6347 \div 2\frac{3}{4}$, to their simplest forms.

140. Find the cube root of 884736, and the fourth root of $95951\frac{111}{625}$.

141. A general levied a contribution of $520 on four villages, containing 250, 300, 400, and 500 inhabitants respectively: what must they each pay?

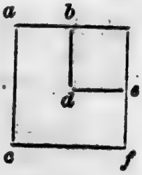
142. A person had a salary of $520 a year, and let it remain unpaid for 17 years. How much had he to receive at the end of that time, allowing 6 per cent. per annum compound interest, payable half-yearly?

143. Insert four arithmetical means between 2 and 79; also find the 9th term and the sum of the first 207 terms of the series 3, 7, 11, 15, &c.

144. A, B, and C, start at the same time, from the same point, and in the same direction, round an island 73 miles in circumference; A goes at the rate of 6, B at the rate of 10, and C at the rate of 16 miles per day. In what time will they be all together again?

ARITHMETICAL RECREATIONS.

1. If the third of 6 be 3 what must the fourth of 20 be?
2. If the half of 5 be 7 what part of 9 will be 11?
3. Place four *nines* so that their sum shall be 100.
4. What part of 8 pence is the third of two pence?
5. If a herring and a half cost 1½d., how much will 11 herrings cost?
6. If 12 apples are worth 21 pears, and 8 pears cost a cent, what will be the price of 100 apples?
7. Find a number such that 5 shall be the three-sevenths of it.
8. A hundred hurdles are so placed as to inclose 200 sheep, and with two hurdles more the field may be made to hold 400; how is this to be done?
9. A gentleman who owned four hundred acres of land in the form of a square, desired to keep 100 acres, also in the form of a square, in one corner, and divide the remainder, *a b c d e f*, equally among his four sons, so that each son should have his lot of the same shape as his brother's. How may this be done?



10. Place four *threes* so as to make 34.
11. Write down 13 in such a way that rubbing half of it out 8 shall remain.
12. Two thirsty persons cast away on a desert island, find an 8 gallon cask of water. They wish to divide it equally between them, but have no other measures than the 8 gallon cask, a five gallon cask and a three gallon cask. How can they divide it?
13. How must a board 16 inches long and 9 inches wide be cut into two such parts, that when they are joined together they may form a square?
14. Place the 9 digits in the accompanying figure, one digit to each division, in such a way that when added vertically, horizontally or diagonally, the sum shall always be the same.

15. Three persons bought a quantity of sugar weighing 51 lbs., and wish to part it equally between them. They have no weights but a 4 lb. weight and a 7 lb. weight. How can they divide it?

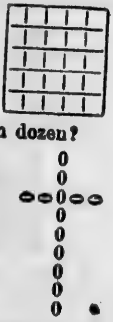
16. Suppose 26 hurdles can be placed in a rectangular form so as to inclose 40 square yards of ground; how can they be placed when two of them are taken away, so as to inclose 120 square yards?

17. A person has a fox, a goose, and a peck of oats to carry over a river, but on account of the smallness of the boat he can only carry over one at a time. How can this be done so as not to leave the fox with the goose, nor the goose with the oats?

18. In a distant village of Canada, there was stationed a small detachment of troops consisting of a sergeant and 24 men. Having constructed temporary barracks, the sergeant divided them into 9 compartments, allotting the centre one to himself, and the rest to his men. One evening the sergeant, wishing to ascertain if all were in, visited each compartment, and finding 3 men in each, making 9 in each row, retired. Four men, however, went out, and the sergeant feeling shortly afterwards uneasy, returned to count his men, but still finding 9 in each row, retired again; the 4 men then came back, bringing each another man with him, and the sergeant upon going his round once more, counted as before, and retired perfectly satisfied. After he left, four more men were introduced, and once more the sergeant, entertaining a suspicion that all was not right, counted, but finding the number still the same in each row, he left. No sooner had he left, than four more men came in, making 12 strangers; and once more the sergeant inspected the compartments to his satisfaction. Finally the 12 strangers left, taking with them 6 of the soldiers, and the sergeant counting once more retired to rest, persuaded that no one had gone out or come in, and that his suspicions were unfounded. How was this possible?

19. Write down 12 so that by rubbing out one half 7 shall remain.

20. Place the first 25 numbers, 1, 2, 3, 4, 5, &c., in the divisions of the accompanying figure, so that the columns added in any order, i. e., upwards, horizontally, or diagonally, may amount to the same sum.



21. What is the difference between half-a-dozen dozen and six dozen dozen?

22. If a cross be made of 13 counters as in the margin, *nine* may be reckoned in three ways, i. e., by counting from the bottom up to the top of the perpendicular line; from the bottom up to the cross and then to the right; or from the bottom up to the cross and then to the left. Now take away two of the counters and with the others form a cross which shall possess the same property of counting *nine* when thus reckoned.

0
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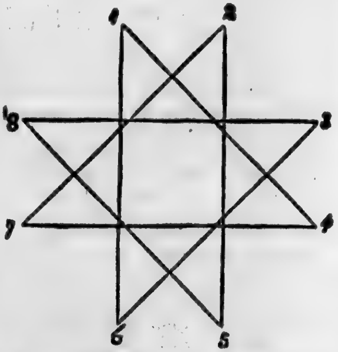
23. Seven out of 21 bottles being full of wine, 7 half full and 7 empty—it is required to distribute them among 3 persons, so that each may have the same quantity of wine and the same number of bottles.

24. Two travellers, one of whom had with him 5 bottles of wine and the other 3, were joined by a third person, who, after the wine was drunk, left 8 shillings for his just share of it; how is this to be divided between the other two?

25. A person having by accident broken a basket of eggs, offered to pay for them on the spot if the owner could tell how many he had; to which he replied that he only knew there were between 50 and 100, and that when he counted them by 2's and 3's at a time none remained; but when he counted them by 5 at a time there were 3 remaining; how many eggs had he?

26. It is required to find 4 such weights that they weigh any number of pounds from 1 to 40.

27. In the accompanying figure it is required to fill seven out of the eight points with counters in the following manner, i. e., the counter has to start from an *unoccupied* point, pass along the line and be deposited at the other extremity. Thus, in commencing, the counter may start from any point, since all are unoccupied, starting from 1 the counter may be carried either to 6 or to 4 and there deposited, suppose it to be deposited at 6, then the next counter may start from any point except 6, and so on.



28. A brazen lion, placed in the middle of a reservoir, throws out water from its mouth, its eyes and its right foot. When the water flows from its mouth alone, it fills the reservoir in 6 hours; from the right eye it fills it in 2 days; from the left eye in 3 days, and from the foot in 4 days. In what time will the basin be filled by the water flowing from all these apertures at once?

29. Desire a person to think of any three numbers, each less than 10, and then tell him the numbers thought of.

30. Three men, Jones, Brown, and Smith, with their sons Harry, Tom, and Ned, had each a piece of land in the form of a square. Jones' piece was 23 rods longer on each side than Tom's, and Brown's piece was 11 rods longer on each side than Harry's. Each man possessed 63 square rods of land more than his son. Which of the persons were father and son respectively?

31. A sea-captain, on a voyage, had a crew of 30 men, half of whom were blacks. Being becalmed on the passage for a long time, their provisions began to fail, and the captain became satisfied that, unless the number of men were greatly diminished, all would perish of hunger before they could reach any friendly port. He therefore proposed to the sailors that they should stand in a row on deck, and that every ninth man should be thrown overboard, until one-half of the crew were thus destroyed. To this they all agreed. How should they stand so as to save the whites?

32. Direct a person to multiply together two numbers, one of which you select, and, unseen by you, to rub out one of the digits of the product—it is required to tell, upon his reading the remaining digits of the product, what figure was rubbed out.

33. It is required to write down beforehand the answer to a question in addition of a given number of lines, you writing the *second*, *fourth*, *sixth*, &c., addends, and some other person the intermediate ones.

MATHEMATICAL TABLES.

LOGARITHMS OF NUMBERS FROM 1 TO 10,000, WITH DIFFERENCES AND PROPORTIONAL PARTS.

Numbers from 1 to 100.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0·000000	21	1·322219	41	1·612784	61	1·785330	81	1·908485
2	0·301030	22	1·342423	42	1·623249	62	1·792392	82	1·913814
3	0·477121	23	1·361728	43	1·633468	63	1·799341	83	1·919078
4	0·602060	24	1·380211	44	1·643453	64	1·806180	84	1·924279
5	0·698970	25	1·397940	45	1·653213	65	1·812913	85	1·929419
6	0·778151	26	1·414973	46	1·662758	66	1·819544	86	1·934[illegible]
7	0·845098	27	1·431364	47	1·672098	67	1·826075	87	1·939519
8	0·903090	28	1·447158	48	1·681241	68	1·832509	88	1·944483
9	0·954243	29	1·462398	49	1·690196	69	1·838849	89	1·949390
10	1·000000	30	1·477121	50	1·698970	70	1·845098	90	1·954243
11	1·041393	31	1·491362	51	1·707570	71	1·851258	91	1·959041
12	1·079181	32	1·505150	52	1·716003	72	1·857332	92	1·963788
13	1·113943	33	1·518514	53	1·724276	73	1·863323	93	1·968483
14	1·146128	34	1·531479	54	1·732394	74	1·869232	94	1·973128
15	1·176091	35	1·544068	55	1·740363	75	1·875061	95	1·977724
16	1·204120	36	1·556303	56	1·748188	76	1·880814	96	1·982271
17	1·230449	37	1·568202	57	1·755875	77	1·886491	97	1·986772
18	1·255273	38	1·579784	58	1·763428	78	1·892095	98	1·991226
19	1·278754	39	1·591065	59	1·770852	79	1·897627	99	1·995635
20	1·301030	40	1·602060	60	1·778151	80	1·903090	100	2·000000

PP	N.	0	1	2	3	4	5	6	7	8	9	D.
	100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
41	1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
83	2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
124	3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
166	4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
207	5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
248	6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
290	7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
331	8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
373	9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
	110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
38	1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
76	2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
113	3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
151	4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
189	5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
227	6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
265	7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
302	8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
340	9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
	120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
35	1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
70	2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
104	3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
139	4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
174	5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
209	6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
244	7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
278	8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253	338
313	9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
	130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
32	1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
64	2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
97	3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
129	4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
161	5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
193	6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
225	7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
258	8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702	314
290	9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
	140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911	309
30	1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982	307
60	2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032	305
90	3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
120	4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068	301
150	5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055	299
180	6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
210	7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
240	8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
270	9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
	150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689	289
28	1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558	287
56	2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407	285
84	3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
112	4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051	281
140	5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846	279
168	6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
196	7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
224	8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
252	9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848	273

PP	N.	0	1	2	3	4	5	6	7	8	9	D.
	160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
26	1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
53	2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
79	3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
106	4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
132	5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
158	6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
184	7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
210	8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
237	9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
	170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	254
25	1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
50	2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
74	3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
99	4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
124	5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
149	6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
174	7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
198	8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
223	9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
	180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
24	1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
47	2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
71	3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
94	4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
118	5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
141	6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
165	7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
188	8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
212	9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
	190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
22	1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
45	2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
67	3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
89	4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
112	5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
134	6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
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178	8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
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	200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
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191	9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012	207
	210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077	206
20	1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
40	2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
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101	5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
121	6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6252	201
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162	8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246	199
182	9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225	198

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19	1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
39	2	6353	6549	6744	6939	7135	7330	7525	7720	7915	81[illegible]	195
58	3	8305	8500	8694	8889	9083	9278	9472	9666	9860	350054	194
77	4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989	193
97	5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
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	230	361728	361917	362105	362294	362482	362671	362859	363048	363236	363424	189
19	1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
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56	3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
74	4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
93	5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	184
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167	9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030	181
	240	380211	380392	380573	380754	380934	381115	381296	381476	381656	381837	181
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35	2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
53	3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
71	4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
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	250	397940	398114	398287	398461	398634	398808	398981	399154	399328	399501	173
17	1	9674	9847	400020	400192	400365	400538	400711	400883	401056	401228	173
34	2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	172
51	3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
68	4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
85	5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
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136	8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
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	260	414973	415140	415307	415474	415641	415808	415974	416141	416308	416474	167
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49	3	9956	420121	420286	420451	420616	420781	420945	421110	421275	421439	165
66	4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
82	5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
98	6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
115	7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
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148	9	9752	9914	430075	430236	430398	430559	430720	430881	431042	431203	161
	270	431364	431525	431685	431846	432007	432167	432328	432488	432649	432809	161
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32	2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
47	3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
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79	5	9333	9491	9648	9806	9964	440122	440279	440437	440594	440752	158
95	6	440909	441066	441224	441381	441538	1695	1852	2009	2166	2323	157
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31	2	450249	450403	450557	450711	450865	451018	451172	451326	451479	1633	154
46	3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
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77	5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
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15	1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
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74	5	9822	9969	470116	470263	470410	470557	470704	470851	470998	471145	147
88	6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
103	7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
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	300	477121	477266	477411	477555	477700	477844	477989	478133	478278	478422	145
14	1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
29	2	480007	480151	480294	480438	480582	480725	480869	481012	481156	481299	144
43	3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
57	4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
72	5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
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129	9	9958	490099	490239	490380	490520	490661	490801	490941	491081	491222	140
	310	491362	491502	491642	491782	491922	492062	492201	492341	492481	492621	140
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28	2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
41	3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
55	4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
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83	6	9687	9824	9962	500099	500236	500374	500511	500648	500785	500922	137
97	7	501059	501196	501333	1470	1607	1744	1880	2017	2154	2291	137
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	320	505150	505286	505421	505557	505693	505828	505964	506099	506234	506370	136
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54	4	510545	510679	510813	510947	511081	511215	1349	1482	1616	1750	134
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94	7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
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39	3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
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78	6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
91	7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
104	8	8917	9045	9174	9302	9430	9559	9687	9815	9943	530072	128
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38	3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
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63	5	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
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	350	544068	544192	544316	544440	544564	544688	544812	544936	545060	545183	124
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24	2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
37	3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
49	4	9003	9126	9249	9371	9494	9616	9739	9861	9984	550106	123
61	5	550228	550351	550473	550595	550717	550840	550962	551084	551206	1328	122
73	6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
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48	4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
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	370	568202	568319	568436	568554	568671	568788	568905	569023	569140	569257	117
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46	4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
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	390	591065	591176	591287	591399	591510	591621	591732	591843	591955	592066	111
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77	7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
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21	2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
32	3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
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54	5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
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75	7	9594	9701	9808	9914	610021	610128	610234	610341	610447	610554	107
86	8	610660	610767	610873	610979	1086	1192	1298	1405	1511	1617	106
96	9	1723	1829	1935	2042	2148	2254	2360	2466	2572	2678	106
	410	612784	312890	612996	613102	613207	613313	613419	613525	613630	613736	106
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21	2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
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42	4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
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63	6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
74	7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072	104
84	8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
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	420	323249	623353	623456	623559	623663	623766	623869	623973	624076	624179	103
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31	3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
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92	9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
	430	633468	633569	633670	633771	633872	633973	634074	634175	634276	634376	101
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29	3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
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49	5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
59	6	9335	9432	9530	9627	9724	9821	649919	650016	650113	650210	97
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88	9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
	450	653213	553309	653405	653502	653598	653695	653791	653888	653984	654080	96
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56	6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
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	470	672098	672190	672283	672375	672467	672560	672652	672744	672836	672929	92
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28	3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
37	4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
46	5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
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	490	690196	690285	690373	690462	690550	690639	690728	690816	690905	690993	89
9	1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
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35	4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
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59	7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
67	8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
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25	3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
33	4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
41	5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
50	6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	83
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	530	724276	724358	724440	724522	724604	724685	724767	724849	724931	725013	82
8	1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
16	2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
24	3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
32	4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
41	5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
49	6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
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65	8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
73	9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
	540	732394	732474	732555	732635	732715	732796	732876	732956	733037	733117	80
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48	6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
56	7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
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	550	740363	740442	740521	740600	740678	740757	740836	740915	740994	741073	79
8	1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
16	2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
23	3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	73
31	4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
39	5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
47	6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
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	560	748188	748266	748343	748421	748498	748576	748653	748731	748808	748885	77
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15	2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
23	3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
31	4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
39	5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
46	6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
54	7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
62	8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
69	9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
	570	755875	755951	756027	756103	756180	756256	756332	756408	756484	756560	76
8	1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
15	2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
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30	4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
38	5	9668	9743	9819	9894	9970	760045	760121	760196	760272	760347	76
46	6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
53	7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
61	8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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	580	763428	763503	763578	763653	763727	763802	763877	763952	764027	764101	75
7	1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
15	2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
22	3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
30	4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
37	5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
44	6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
52	7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
59	8	9377	9451	9525	9599	9673	9746	9620	9894	9968	770042	74
67	9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
	590	770852	770926	770999	771073	771146	771220	771293	771367	771440	771514	74
7	1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
15	2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
22	3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
29	4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
37	5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
44	6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
51	7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
58	8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
66	9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
	600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
7	1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
14	2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
22	3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
29	4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
36	5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
43	6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
50	7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
58	8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
65	9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
	610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970	71
7	1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
14	2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
21	3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
28	4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
36	5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
43	6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
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57	8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
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	620	792392	792462	792532	792602	792672	792742	792812	792882	792952	793022	70
7	1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
14	2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
21	3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
28	4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
35	5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
42	6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
49	7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
56	8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
63	9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
	630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961	69
7	1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
14	2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
21	3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
28	4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
35	5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
41	6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
48	7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
55	8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
62	9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68

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7	1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
13	2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
20	3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
27	4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
34	5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
40	6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
47	7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
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	650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
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13	2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
20	3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
26	4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
33	5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
40	6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
46	7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
53	8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
59	9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
	660	819544	819610	819676	819741	819807	819873	819939	820004	820070	820136	66
7	1	820201	820267	820333	820399	820464	820530	820595	0661	0727	0792	66
13	2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
20	3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
26	4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
33	5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
39	6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
46	7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
52	8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
59	9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
	670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
6	1	6723	6787	6852	6917	6981	7043	7111	7175	7240	7305	65
13	2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
19	3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
26	4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
32	5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
38	6	9947	830011	830075	830139	830204	830268	830332	830396	830460	830525	64
45	7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
51	8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
58	9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
	680	832509	832573	832637	832700	832764	832828	832892	832956	833020	833083	64
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13	2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
19	3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
25	4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
32	5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
38	6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
44	7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
50	8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
57	9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
	690	838849	838912	838975	839038	839101	839164	839227	839289	839352	839415	63
6	1	9478	9541	9604	9667	9729	9792	9855	9918	9981	840043	63
13	2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	63
19	3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
25	4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
32	5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
38	6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
44	7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
50	8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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6	1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
12	2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
19	3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
25	4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
31	5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
37	6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
43	7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
50	8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
56	9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
	710	851258	851320	851381	851442	851503	851564	851625	851686	851747	851809	61
6	1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
12	2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
18	3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
24	4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
31	5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
37	6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
43	7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
49	8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
55	9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
	720	857332	857393	857453	857513	857574	857634	857694	857755	857815	857875	60
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12	2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
18	3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
24	4	9739	9799	9859	9918	9978	860038	860098	860158	860218	860278	60
30	5	860338	860398	860458	860518	860578	0637	0697	0757	0817	0877	60
36	6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
42	7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
48	8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
54	9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
	730	863323	863382	863442	863501	863561	863620	863680	863739	863799	863858	59
6	1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
12	2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
18	3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
24	4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
30	5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
35	6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
41	7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
47	8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
53	9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
	740	869232	869290	869349	869408	869466	869525	869584	869642	869701	869760	59
6	1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
12	2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930	58
17	3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
23	4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
29	5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
35	6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
41	7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
46	8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
52	9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
	750	875061	875119	875177	875235	875293	875351	875409	875466	875524	875582	58
6	1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
12	2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
17	3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
23	4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
29	5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
35	6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
41	7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
46	8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
52	9	880242	880299	880356	880413	880471	880528	0585	0642	0699	0756	57

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	760	880814	880871	880928	880985	881042	881099	881156	881213	881271	881328	57
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11	2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
17	3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
23	4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
29	5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
34	6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
40	7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
46	8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
51	9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
	770	886491	886547	886604	886660	886716	886773	886829	886885	886942	886998	56
6	1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
11	2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
17	3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
22	4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
28	5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
34	6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
39	7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
45	8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
50	9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
	780	892095	892150	892206	892262	892317	892373	892429	892484	892540	892595	56
6	1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
11	2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
17	3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
22	4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
27	5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
33	6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
38	7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
44	8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
49	9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
	790	897627	897682	897737	897792	897847	897902	897957	898012	898067	898122	55
5	1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
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17	3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
22	4	9821	9875	9930	9985	900039	900094	900149	900203	900258	900312	55
27	5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859	55
33	6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
38	7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1949	54
44	8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
49	9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
	800	903090	903144	903199	903253	903307	903361	903416	903470	903524	903578	54
5	1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
11	2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
16	3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
22	4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
27	5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
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38	7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
43	8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
49	9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
	810	908485	908539	908592	908646	908699	908753	908807	908860	908914	908967	54
5	1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
11	2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037	53
16	3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
21	4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
27	5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
32	6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
37	7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
42	8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
48	9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53

PP	N.	0	1	2	3	4	5	6	7	8	9	D.
	820	913814	913867	913920	913973	914026	914079	914132	914184	914237	914290	53
5	1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
11	2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
16	3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
21	4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
27	5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
32	6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
37	7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
42	8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
48	9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
	830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549	52
5	1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
10	2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
16	3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
21	4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
26	5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
31	6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
36	7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
42	8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
47	9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
	840	924279	924331	924383	924434	924486	924538	924589	924641	924693	924744	52
5	1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
10	2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
15	3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
20	4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
26	5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
31	6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
36	7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
41	8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
46	9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
	850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879	51
5	1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389	51
10	2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898	51
15	3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
20	4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
26	5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
31	6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
36	7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
41	8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
46	9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
	860	934498	934549	934599	934650	934700	934751	934801	934852	934902	934953	50
5	1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
10	2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
15	3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
20	4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
25	5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
30	6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
35	7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
40	8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
45	9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
	870	939519	939569	939619	939669	939719	939769	939819	939869	939918	939968	50
5	1	940018	940068	940118	940168	940218	940267	940317	940367	940417	940467	50
10	2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
15	3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
20	4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
25	5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
30	6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
35	7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
40	8	3495	3544	3593	3643	3692	3742	3792	3841	3890	3939	49
45	9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49

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	880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927	49
5	1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
10	2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
15	3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
20	4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
25	5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
29	6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
34	7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
39	8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
44	9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
	890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829	49
5	1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316	49
10	2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
15	3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
20	4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
24	5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
29	6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
34	7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
39	8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
44	9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
	900	954243	954291	954339	954387	954435	954484	954532	954580	954628	954677	48
5	1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
10	2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
14	3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
19	4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
24	5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
29	6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
34	7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
38	8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
43	9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
	910	959041	959089	959137	959185	959232	959280	959328	959375	959423	959471	48
5	1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
9	2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
14	3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
19	4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
24	5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
28	6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
33	7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
38	8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
42	9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
	920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212	47
5	1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
9	2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
14	3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
19	4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
23	5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
28	6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
33	7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
38	8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
42	9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
	930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903	47
5	1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
9	2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
14	3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300	47
18	4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765	46
23	5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
28	6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
32	7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
37	8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
41	9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46

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	940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543	46
5	1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
9	2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
14	3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
18	4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
23	5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
28	6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
32	7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
37	8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
41	9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
	950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
5	1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
9	2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
14	3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
18	4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
23	5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
27	6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
32	7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
36	8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
41	9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
	960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
5	1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
9	2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
14	3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
18	4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
23	5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
27	6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
32	7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
36	8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
41	9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
	970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
5	1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
9	2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
14	3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
18	4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
23	5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
27	6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
32	7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
36	8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
41	9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
	980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
4	1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
9	2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
13	3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
18	4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
22	5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
26	6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
31	7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
35	8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
40	9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
	990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
4	1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
9	2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
13	3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
18	4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
22	5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
26	6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
31	7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
35	8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
40	9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
1	1	1	1·0000000	1·000000	64	4096	262144	8·0000000	4·000000
2	4	8	1·4142136	1·259921	65	4225	274625	8·0622577	4·020726
3	9	27	1·7320508	1·442250	66	4356	287496	8·1240384	4·041240
4	16	64	2·0000000	1·587401	67	4489	300763	8·1853528	4·061548
5	25	125	2·2360680	1·709976	68	4624	314432	8·2462113	4·081656
6	36	216	2·4494897	1·817121	69	4761	328509	8·3066239	4·101566
7	49	343	2·6457513	1·912931	70	4900	343000	8·3666003	4·121285
8	64	512	2·8284271	2·000000	71	5041	357911	8·4261498	4·140818
9	81	729	3·0000000	2·080084	72	5184	373248	8·4852814	4·160168
10	100	1000	3·1622777	2·154435	73	5329	389017	8·5440037	4·179339
11	121	1331	3·3166248	2·223980	74	5476	405224	8·6023253	4·198336
12	144	1728	3·4641016	2·289428	75	5625	421875	8·6602540	4·217163
13	169	2197	3·6055513	2·351335	76	5776	438976	8·7177979	4·235824
14	196	2744	3·7416574	2·410142	77	5929	456533	8·7749644	4·254321
15	225	3375	3·8729833	2·466212	78	6084	474552	8·8317609	4·272659
16	256	4096	4·0000000	2·519842	79	6241	493039	8·8881944	4·290841
17	289	4913	4·1231056	2·571282	80	6400	512000	8·9442719	4·308870
18	324	5832	4·2426407	2·620741	81	6561	531441	9·0000000	4·326749
19	361	6859	4·3588989	2·668402	82	6724	551368	9·0553851	4·344481
20	400	8000	4·4721360	2·714418	83	6889	571787	9·1104336	4·362071
21	441	9261	4·5825757	2·758924	84	7056	592704	9·1651514	4·379519
22	484	10648	4·6904158	2·802039	85	7225	614125	9·2195445	4·396830
23	529	12167	4·7958315	2·843867	86	7396	636056	9·2736185	4·414005
24	576	13824	4·8989795	2·884499	87	7569	658503	9·3273791	4·431047
25	625	15625	5·0000000	2·924018	88	7744	681472	9·3808315	4·447960
26	676	17576	5·0990195	2·962496	89	7921	704969	9·4339811	4·464745
27	729	19683	5·1961524	3·000000	90	8100	729000	9·4868330	4·481405
28	784	21952	5·2915026	3·036589	91	8281	753571	9·5393920	4·497941
29	841	24389	5·3851648	3·072317	92	8464	778688	9·5916630	4·514357
30	900	27000	5·4772256	3·107232	93	8649	804357	9·6436508	4·530655
31	961	29791	5·5677644	3·141381	94	8836	830584	9·6953597	4·546836
32	1024	32768	5·6568542	3·174802	95	9025	857375	9·7467943	4·562903
33	1089	35937	5·7445626	3·207534	96	9216	884736	9·7979590	4·578857
34	1156	39304	5·8309519	3·239612	97	9409	912673	9·8488578	4·594701
35	1225	42875	5·9160798	3·271066	98	9604	941192	9·8994949	4·610436
36	1296	46656	6·0000000	3·301927	99	9801	970299	9·9498744	4·626065
37	1369	50653	6·0827625	3·332222	100	10000	1000000	10·0000000	4·641589
38	1444	54872	6·1644140	3·361975	101	10201	1030301	10·0498756	4·657010
39	1521	59319	6·2449980	3·391211	102	10404	1061208	10·0995049	4·672329
40	1600	64000	6·3245553	3·419952	103	10609	1092727	10·1488916	4·687548
41	1681	68921	6·4031242	3·448217	104	10816	1124864	10·1980390	4·702669
42	1764	74088	6·4807407	3·476027	105	11025	1157625	10·2469508	4·717694
43	1849	79507	6·5574385	3·503398	106	11236	1191016	10·2956301	4·732624
44	1936	85184	6·6332496	3·530348	107	11449	1225043	10·3440804	4·747459
45	2025	91125	6·7082039	3·556893	108	11664	1259712	10·3923048	4·762203
46	2116	97336	6·7823300	3·583048	109	11881	1295029	10·4403065	4·776856
47	2209	103823	6·8556546	3·608826	110	12100	1331000	10·4880885	4·791420
48	2304	110592	6·9282032	3·634241	111	12321	1367631	10·5356538	4·805896
49	2401	117649	7·0000000	3·659306	112	12544	1404928	10·5830052	4·820284
50	2500	125000	7·0710678	3·684031	113	12769	1442897	10·6301458	4·834588
51	2601	132651	7·1414284	3·708430	114	12996	1481544	10·6770783	4·848808
52	2704	140608	7·2111026	3·732511	115	13225	1520875	10·7238053	4·862944
53	2809	148877	7·2801099	3·756286	116	13456	1560896	10·7703296	4·876999
54	2916	157464	7·3484692	3·779763	117	13689	1601613	10·8166538	4·890973
55	3025	166375	7·4161985	3·802953	118	13924	1643032	10·8627805	4·904868
56	3136	175616	7·4833148	3·825862	119	14161	1685159	10·9087121	4·918685
57	3249	185193	7·5498344	3·848501	120	14400	1728000	10·9544512	4·932424
58	3364	195112	7·6157731	3·870877	121	14641	1771561	11·0000000	4·946088
59	3481	205379	7·6811457	3·892996	122	14884	1815848	11·0453610	4·959675
60	3600	216000	7·7459667	3·914867	123	15129	1860867	11·0905365	4·973190
61	3721	226981	7·8102497	3·936497	124	15376	1906624	11·1355287	4·986631
62	3844	238328	7·8740079	3·957892	125	15625	1953125	11·1803399	5·000000
63	3969	250047	7·9372539	3·979057	126	15876	2000376	11·2249722	5·013298

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
127	16129	2048383	11·2694277	5·026526	190	36100	6859000	13·7840488	5·748897
128	16384	2097152	11·3137085	5·039684	191	36481	6967871	13·8202750	5·758965
129	16641	2146689	11·3578167	5·052774	192	36864	7077888	13·8564065	5·768998
130	16900	2197000	11·4017543	5·065797	193	37249	7189057	13·8924440	5·778996
131	17161	2248091	11·4455231	5·078753	194	37636	7301384	13·9283883	5·788960
132	17424	2299968	11·4891253	5·091643	195	38025	7414875	13·9642400	5·798890
133	17689	2352637	11·5325626	5·104469	196	38416	7529536	14·0000000	5·808786
134	17956	2406104	11·5758369	5·117230	197	38809	7645373	14·0356688	5·818272
135	18225	2460375	11·6189500	5·129928	198	39204	7762392	14·0712473	5·828648
136	18496	2515456	11·6619033	5·142563	199	39601	7880599	14·1067360	5·838476
137	18769	2571353	11·7046999	5·155137	200	40000	8000000	14·1421356	5·848035
138	19044	2628072	11·7473444	5·167649	201	40401	8120601	14·1774469	5·857766
139	19321	2685619	11·7898261	5·180101	202	40804	8242408	14·2126704	5·867464
140	19600	2744000	11·8321596	5·192494	203	41209	8365427	14·2478068	5·877130
141	19881	2803221	11·8743421	5·204828	204	41616	8489664	14·2828569	5·886765
142	20164	2863288	11·9163753	5·217103	205	42025	8615125	14·3178211	5·896368
143	20449	2924207	11·9582607	5·229321	206	42436	8741816	14·3527001	5·905941
144	20736	2985984	12·0000000	5·241483	207	42849	8869743	14·3874946	5·915483
145	21025	3048625	12·0415946	5·253588	208	43264	8998912	14·4222051	5·924993
146	21316	3112136	12·0830460	5·265637	209	43681	9123329	14·4568323	5·934473
147	21609	3176523	12·1243557	5·277632	210	44100	9261000	14·4913767	5·943921
148	21904	3241792	12·1655251	5·289572	211	44521	9393931	14·5258390	5·953341
149	22201	3307949	12·2065556	5·301459	212	44944	9528128	14·5602198	5·962731
150	22500	3375000	12·2474487	5·313293	213	45369	9663597	14·5945195	5·972091
151	22801	3442951	12·2882056	5·325074	214	45796	9800344	14·6287388	5·981426
152	23104	3511808	12·3288280	5·336803	215	46225	9938375	14·6628783	5·990727
153	23409	3581577	12·3693169	5·348481	216	46656	10077696	14·6969385	6·000000
154	23716	3652264	12·4096736	5·360108	217	47089	10218313	14·7309199	6·009244
155	24025	3723875	12·4498996	5·371685	218	47524	10360232	14·7648231	6·018463
156	24336	3796416	12·4899960	5·383213	219	47961	10503459	14·7986486	6·027650
157	24649	3869893	12·5299641	5·394691	220	48400	10648000	14·8323970	6·036811
158	24964	3944312	12·5698051	5.406120	221	48841	10793861	14·8660687	6·045943
159	25281	4019679	12·6095202	5·417501	222	49284	10941048	14·8996644	6·055048
160	25600	4096000	12·6491106	5·428835	223	49729	11089567	14·9331845	6·064126
161	25921	4173281	12·6885775	5·440122	224	50176	11239424	14·9666295	6·073178
162	26244	4251528	12·7279221	5·451362	225	50625	11390625	15·0000000	6·082201
163	26569	4330747	12·7671453	5·462556	226	51076	11543176	15·0332964	6·091199
164	26896	4410944	12·8062485	5·473704	227	51529	11697083	15·0665192	6·100170
165	27225	4492125	12·8452326	5·484806	228	51984	11852352	15·0996689	6·109115
166	27556	4574296	12·8840987	5·495865	229	52441	12008989	15·1327460	6·118033
167	27889	4657463	12·9228480	5·506879	230	52900	12167000	15·1657509	6·126925
168	28224	4741632	12·9614814	5·517848	231	53361	12326391	15·1986842	6·135792
169	28561	4826809	13·0000000	5·528775	232	53824	12487168	15·2315462	6·144634
170	28900	4913000	13·0384048	5·539658	233	54289	12649337	15·2643375	6·153449
171	29241	5000211	13·0766968	5·550499	234	54756	12812904	15·2970585	6·162239
172	29584	5088448	13·1148770	5·561298	235	55225	12977875	15·3297097	6·171005
173	29929	5177717	13·1529464	5·572055	236	55696	13144256	15·3622915	6·179747
174	30276	5268024	13·1909060	5·582770	237	56169	13312053	15·3948043	6·188463
175	30625	5359375	13·2287566	5·593445	238	56644	13481272	15·4272486	6·197154
176	30976	5451776	13·2664992	5·604079	239	57121	13651919	15·4596248	6·205822
177	31329	5545233	13·3041347	5·614673	240	57600	13824000	15·4919334	6·214464
178	31684	5639752	13·3416641	5·625226	241	58081	13997521	15·5241747	6·223084
179	32041	5735339	13·3790882	5·635741	242	58564	14172488	15·5563492	6·231679
180	32400	5832000	13·4164079	5·646216	243	59049	14348907	15·5884573	6·240251
181	32761	5929741	13·4536240	5·656651	244	59536	14526789	15·6204994	6·248800
182	33124	6028568	13·4907376	5·667051	245	60025	14706125	15·6524758	6·257324
183	33489	6128487	13·5277493	5·677411	246	60516	14886936	15·6843871	6·265826
184	33856	6229504	13·5646600	5·687734	247	61009	15069223	15·7162336	6·274305
185	34225	6331625	13·6014705	5·698019	248	61504	15252992	15·7480157	6·282760
186	34596	6434856	13·6381817	5·708267	249	62001	15438249	15·7797338	6·291194
187	34969	6539203	13·6747943	5·718479	250	62500	15625000	15·8113883	6·299604
188	35344	6644672	13·7113092	5·728654	251	63001	15813251	15·8429795	6·307993
189	35721	6751269	13·7477271	5·738794	252	63504	16003008	15·8745079	6·316359

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
253	64009	16194277	15·9059737	6·324704	316	99856	31554496	17·7763888	6·811284
254	64516	16387064	15·9373775	6·333026	317	100489	31855013	17·8044938	6·818462
255	65025	16581375	15·9687194	6·341326	318	101124	32157432	17·8325545	6·825624
256	65536	16777216	16·0000000	6·349604	319	101761	32461759	17·8605711	6·832771
257	66049	16974593	16·0312195	6·357861	320	102400	32768000	17·8885438	6·839904
258	66564	17173512	16·0623784	6·366095	321	103041	33076161	17·9164729	6·847021
259	67081	17373979	16·0934769	6·374311	322	103684	33386248	17·9443584	6·854124
260	67600	17576000	16·1245155	6·382504	323	104329	33698267	17·9722008	6·861212
261	68121	17779581	16·1554944	6·390676	324	104976	34012224	18·0000000	6·868285
262	68644	17984728	16·1864141	6·398828	325	105625	34328125	18·0277564	6·875344
263	69169	18191447	16·2172747	6·406958	326	106276	34645976	18·0554701	6·882389
264	69696	18399744	16·2480768	6·415068	327	106929	34965783	18·0831413	6·889419
265	70225	18609625	16·2788206	6·423158	328	107584	35287552	18·1107703	6·896435
266	70756	18821096	16·3095064	6·431228	329	108241	35611289	18·1383571	6·903436
267	71289	19034163	16·3401346	6·439277	330	108900	35937000	18·1659021	6·910423
268	71824	19248832	16·3707055	6·447305	331	109561	36264691	18·1934054	6·917396
269	72361	19465109	16·4012195	6·455315	332	110224	36594368	18·2208672	6·924355
270	72900	19683000	16·4316767	6·463304	333	110889	36926037	18·2482876	6·931301
271	73441	19902511	16·4620776	6·471274	334	111556	37259704	18·2756669	6·938232
272	73984	20123648	16·4924225	6·479224	335	112225	37595375	18·3030052	6·945149
273	74529	20346417	16·5227116	6·487154	336	112896	37933056	18·3303028	6·952053
274	75076	20570824	16·5529454	6·495065	337	113569	38272753	18·3575598	6·958943
275	75625	20796875	16·5831240	6·502956	338	114244	38614472	18·3847763	6·965819
276	76176	21024576	16·6132477	6·510830	339	114921	38958219	18·4119526	6·972683
277	76729	21253933	16·6433170	6·518684	340	115600	39304000	18·4390889	6·979532
278	77284	21484952	16·6733320	6·526519	341	116281	39651821	18·4661853	6·986368
279	77841	21717639	16·7032931	6·534335	342	116964	40001688	18·4932420	6·993191
280	78400	21952000	16·7332005	6·542133	343	117649	40353607	18·5202592	7·000000
281	78961	22188041	16·7630546	6·549912	344	118336	40707584	18·5472370	7·006796
282	79524	22425768	16·7928556	6·557672	345	119025	41063625	18·5741756	7·013579
283	80089	22665187	16·8226038	6·565415	346	119716	41421736	18·6010752	7·020349
284	80656	22906304	16·8522995	6·573139	347	120409	41781923	18·6279360	7·027106
285	81225	23149125	16·8819430	6·580844	348	121104	42144192	18·6547581	7·033850
286	81796	23393656	16·9115345	6·588532	349	121801	42508549	18·6815417	7·040581
287	82369	23639903	16·9410743	6·596202	350	122500	42875000	18·7082869	7·047299
288	82944	23887872	16·9705627	6·603854	351	123201	43243551	18·7349940	7·054004
289	83521	24137569	17·0000000	6·611489	352	123904	43614208	18·7616630	7·060696
290	84100	24389000	17·0293864	6·619106	353	124609	43986977	18·7882942	7·067376
291	84681	24642171	17·0587221	6·626705	354	125316	44361864	18·8148877	7·074044
292	85264	24897088	17·0880075	6·634287	355	126025	44738875	18·8414437	7·080699
293	85849	25153757	17·1172428	6·641852	356	126736	45118016	18·8679623	7·087341
294	86436	25412184	17·1464282	6·649399	357	127449	45499293	18·8944436	7·093971
295	87025	25672375	17·1755640	6·656930	358	128164	45882712	18·9208879	7·100588
296	87616	25934336	17·2046505	6·664444	359	128881	46268279	18·9472953	7·107194
297	88209	26198073	17·2336879	6·671940	360	129600	46656000	18·9736660	7·113786
298	88804	26463592	17·2626762	6·679420	361	130321	47045881	19·0000000	7·120367
299	89401	26730899	17·2916165	6·686882	362	131044	47437928	19·0262976	7·126936
300	90000	27000000	17·3205081	6·694329	363	131769	47832147	19·0525589	7·133492
301	90601	27270901	17·3493516	6·701759	364	132496	48228544	19·0787840	7·140037
302	91204	27543608	17·3781472	6·709173	365	133225	48627125	19·1049732	7·146569
303	91809	27818127	17·4068952	6·716570	366	133956	49027896	19·1311265	7·153090
304	92416	28094464	17·4355958	6·723951	367	134689	49430863	19·1572441	7·159599
305	93025	28372625	17·4642492	6·731316	368	135424	49836032	19·1833261	7·166096
306	93636	28652616	17·4928557	6·738665	369	136161	50243409	19·2093727	7·172580
307	94249	28934443	17·5214155	6·745997	370	136900	50653000	19·2353841	7·179054
308	94864	29218112	17·5499288	6·753313	371	137641	51064811	19·2613603	7·185516
309	95481	29503629	17·5783958	6·760614	372	138384	51478848	19·2873015	7·191966
310	96100	29791000	17·6068169	6·767899	373	139129	51895117	19·3132079	7·198405
311	96721	30080231	17·6351921	6·775169	374	139876	52313624	19·3390796	7·204832
312	97344	30371328	17·6635217	6·782423	375	140625	52734375	19·3649167	7·211248
313	97969	30664297	17·6918060	6·789661	376	141376	53157376	19·3907194	7·217652
314	98596	30959144	17·7200451	6·796884	377	142129	53582633	19·4164878	7·224045
315	99225	31255875	17·7482393	6·804092	378	142884	54010152	19·4422221	7·230427

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
379	143641	54439939	19·4679223	7·236797	442	195364	86350888	21·0237960	7·617412
380	144400	54872000	19·4935887	7·243156	443	196249	86938307	21·0475652	7·623152
381	145161	55306341	19·5192213	7·249504	444	197136	87528384	21·0713075	7·628884
382	145924	55742968	19·5448203	7·255841	445	198025	88121125	21·0950231	7·634607
383	146689	56181887	19·5703858	7·262167	446	198916	88716536	21·1187121	7·640321
384	147456	56623104	19·5959179	7·268482	447	199809	89314623	21·1423745	7·646027
385	148225	57066625	19·6214169	7·274786	448	200704	89915392	21·1660105	7·651725
386	148996	57512456	19·6468827	7·281079	449	201601	90518849	21·1896201	7·657414
387	149769	57960603	19·6723156	7·287362	450	202500	91125000	21·2132034	7·663094
388	150544	58411072	19·6977156	7·293633	451	203401	91733851	21·2367606	7·668766
389	151321	58863869	19·7230829	7·299894	452	204304	92345408	21·2602916	7·674430
390	152100	59319000	19·7484177	7·306143	453	205209	92959677	21·2837967	7·680086
391	152881	59776471	19·7737199	7·312383	454	206116	93576664	21·3072758	7·685733
392	153664	60236288	19·7989899	7·318611	455	207025	94196375	21·3307290	7·691372
393	154449	60698457	19·8242276	7·324829	456	207936	94818816	21·3541565	7·697002
394	155236	61162984	19·8494332	7·331037	457	208849	95443993	21·3775583	7·702625
395	156025	61629875	19·8746069	7·337234	458	209764	96071912	21·4009346	7·708239
396	156816	62099136	19·8997487	7·343420	459	210681	96702579	21·4242853	7·713845
397	157609	62570773	19·9248588	7·349597	460	211600	97336000	21·4476106	7·719442
398	158404	63044792	19·9499373	7·355762	461	212521	97972181	21·4709106	7·725032
399	159201	63521199	19·9749844	7·361918	462	213444	98611128	21·4941853	7·730614
400	160000	64000000	20·0000000	7·368063	463	214369	99252847	21·5174348	7·736188
401	160801	64481201	20·0249844	7·374198	464	215296	99897344	21·5406592	7·741753
402	161604	64964808	20·0499377	7·380322	465	216225	100544625	21·5638587	7·747311
403	162409	65450827	20·0748599	7·386437	466	217156	101194696	21·5870331	7·752861
404	163216	65939264	20·0997512	7·392542	467	218089	101847563	21·6101828	7·758402
405	164025	66430125	20·1246118	7·398636	468	219024	102503232	21·6333077	7·763936
406	164836	66923416	20·1494417	7·404720	469	219961	103161709	21·6564078	7·769462
407	165649	67419143	20·1742410	7·410795	470	220900	103823000	21·6794834	7·774980
408	166464	67917312	20·1990099	7·416859	471	221841	104487111	21·7025344	7·780490
409	167281	68417929	20·2237484	7·422914	472	222784	105154048	21·7255610	7·785993
410	168100	68921000	20·2484567	7·428959	473	223729	105823817	21·7485632	7·791487
411	168921	69426531	20·2731349	7·434994	474	224676	106496424	21·7715411	7·796974
412	169744	69934528	20·2977831	7·441019	475	225625	107171875	21·7944947	7·802454
413	170569	70444997	20·3224014	7·447034	476	226576	107850176	21·8174242	7·807925
414	171396	70957944	20·3469899	7·453040	477	227529	108531333	21·8403297	7·813389
415	172225	71473375	20·3715488	7·459036	478	228484	109215352	21·8632111	7·818846
416	173056	71991296	20·3960781	7·465022	479	229441	109902239	21·8860686	7·824294
417	173889	72511713	20·4205779	7·470999	480	230400	110592000	21·9089023	7·829735
418	174724	73034632	20·4450483	7·476966	481	231361	111284641	21·9317122	7·835169
419	175561	73560059	20·4694895	7·482924	482	232324	111980168	21·9544984	7·840595
420	176400	74088000	20·4939015	7·488872	483	233289	112678587	21·9772610	7·846013
421	177241	74618461	20·5182845	7·494811	484	234256	113379904	22·0000000	7·851424
422	178084	75151448	20·5426386	7·500741	485	235225	114084125	22·0227155	7·856828
423	178929	75686967	20·5669638	7·506661	486	236196	114791256	22·0454077	7·862224
424	179776	76225024	20·5912603	7·512571	487	237169	115501303	22·0680765	7·867613
425	180625	76765625	20·6155281	7·518473	488	238144	116214272	22·0907220	7·872994
426	181476	77308776	20·6397674	7·524365	489	239121	116930169	22·1133444	7·878368
427	182329	77854483	20·6639783	7·530248	490	240100	117649000	22·1359436	7·883735
428	183184	78402752	20·6881609	7·536121	491	241081	118370771	22·1585198	7·889095
429	184041	78953589	20·7123152	7·541986	492	242064	119095488	22·1810730	7·894447
430	184900	79507000	20·7364414	7·547842	493	243049	119823157	22·2036033	7·899792
431	185761	80062991	20·7605395	7·553688	494	244036	120553784	22·2261108	7·905129
432	186624	80621568	20·7846097	7·559526	495	245025	121287375	22·2485955	7·910460
433	187489	81182737	20·8086520	7·565355	496	246016	122023936	22·2710575	7·915783
434	188356	81746504	20·8326667	7·571174	497	247009	122763473	22·2934968	7·921100
435	189225	82312875	20·8566536	7·576985	498	248004	123505992	22·3159136	7·926408
436	190096	82881856	20·8806130	7·582786	499	249001	124251499	22·3383079	7·931710
437	190969	83453453	20·9045450	7·588579	500	250000	125000000	22·3606798	7·937005
438	191844	84027672	20·9284495	7·594363	501	251001	125751501	22·3830293	7·942293
439	192721	84604519	20·9523268	7·600138	502	252004	126506008	22·4053565	7·947574
440	193600	85184000	20·9761770	7·605905	503	253009	127263527	22·4276615	7·952848
441	194481	85766121	21·0000000	7·611663	504	254016	128024064	22·4499443	7·958114

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
505	255025	128787625	22·4722051	7·963374	568	322624	183250432	23·8327506	8·281635
506	256036	129554216	22·4944438	7·968627	569	323761	184220009	23·8537209	8·286493
507	257049	130323843	22·5166605	7·973873	570	324900	185193000	23·8746728	8·291344
508	258064	131096512	22·5388553	7·979112	571	326041	186169411	23·8956063	8·296190
509	259081	131872229	22·5610283	7·984344	572	327184	187149248	23·9165215	8·301030
510	260100	132651000	22·5831796	7·989570	573	328329	188132517	23·9374184	8·305865
511	261121	133432831	22·6053091	7·994788	574	329476	189119224	23·9582971	8·310694
512	262144	134217728	22·6274170	8·000000	575	330625	190109375	23·9791576	8·315517
513	263169	135005697	22·6495033	8·005205	576	331776	191102976	24·0000000	8·320335
514	264196	135796744	22·6715681	8·010403	577	332929	192100033	24·0208243	8·325147
515	265225	136590875	22·6936114	8·015595	578	334084	193100552	24·0416306	8·329954
516	266256	137388096	22·7156334	8·020779	579	335241	194104539	24·0624188	8·334755
517	267289	138188413	22·7376340	8·025957	580	336400	195112000	24·0831892	8·339551
518	268324	138991832	22·7596134	8·031129	581	337561	196122941	24·1039416	8·344341
519	269361	139798359	22·7815715	8·036293	582	338724	197137368	24·1246762	8·349126
520	270400	140608000	22·8035085	8·041451	583	339889	198155287	24·1453929	8·353905
521	271441	141420761	22·8254244	8·046603	584	341056	199176704	24·1660919	8·358678
522	272484	142236648	22·8473193	8·051748	585	342225	200201625	24·1867732	8·363446
523	273529	143055667	22·8691933	8·056886	586	343396	201230056	24·2074369	8·368209
524	274576	143877824	22·8910463	8·062018	587	344569	202262003	24·2280829	8·372967
525	275625	144703125	22·9128785	8·067143	588	345744	203297472	24·2487113	8·377719
526	276676	145531576	22·9346899	8·072262	589	346921	204336469	24·2693222	8·382465
527	277729	146363183	22·9564806	8·077374	590	348100	205379000	24·2899156	8·387206
528	278784	147197952	22·9782506	8·082480	591	349281	206425071	24·3104916	8·391942
529	279841	148035889	23·0000000	8·087579	592	350464	207474688	24·3310501	8·396673
530	280900	148877000	23·0217289	8·092672	593	351649	208527857	24·3515913	8·401398
531	281961	149721291	23·0434372	8·097759	594	352836	209584584	24·3721152	8·406118
532	283024	150568768	23·0651252	8·102839	595	354025	210644875	24·3926218	8·410833
533	284089	151419437	23·0867928	8·107913	596	355216	211708736	24·4131112	8·415542
534	285156	152273304	23·1084400	8·112980	597	356409	212776173	24·4335834	8·420246
535	286225	153130375	23·1300670	8·118041	598	357604	213847192	24·4540385	8·424945
536	287296	153990656	23·1516738	8·123096	599	358801	214921799	24·4744765	8·429638
537	288369	154854153	23·1732605	8·128145	600	360000	216000000	24·4948974	8·434327
538	289444	155720872	23·1948270	8·133187	601	361201	217081801	24·5153013	8·439010
539	290521	156590819	23·2163735	8·138223	602	362404	218167208	24·5356883	8·443688
540	291600	157464000	23·2379001	8·143253	603	363609	219256227	24·5560583	8·448360
541	292681	158340421	23·2594067	8·148276	604	364816	220348864	24·5764115	8·453028
542	293764	159220088	23·2808935	8·153294	605	366025	221445125	24·5967478	8·457691
543	294849	160103007	23·3023604	8·158305	606	367236	222545016	24·6170673	8·462348
544	295936	160989184	23·3238076	8·163310	607	368449	223648543	24·6373700	8·467000
545	297025	161878625	23·3452351	8·168309	608	369664	224755712	24·6576560	8·471647
546	298116	162771336	23·3666429	8·173302	609	370881	225866529	24·6779254	8·476289
547	299209	163667323	23·3880311	8·178289	610	372100	226981000	24·6981781	8·480926
548	300304	164566592	23·4093998	8·183269	611	373321	228099131	24·7184142	8·485558
549	301401	165469149	23·4307490	8·188244	612	374544	229220928	24·7386338	8·490185
550	302500	166375000	23·4520788	8·193213	613	375769	230346397	24·7588368	8·494806
551	303601	167284151	23·4733892	8·198175	614	376996	231475544	24·7790234	8·499423
552	304704	168196608	23·4946802	8·203132	615	378225	232608375	24·7991935	8·504035
553	305809	169112377	23·5159520	8·208082	616	379456	233744896	24·8193473	8·508642
554	306916	170031464	23·5372046	8·213027	617	380689	234885113	24·8394847	8·513243
555	308025	170953875	23·5584380	8·217965	618	381924	236029032	24·8596058	8·517840
556	309136	171879616	23·5796522	8·222898	619	383161	237176659	24·8797106	8·522432
557	310249	172808693	23·6008474	8·227825	620	384400	238328000	24·8997992	8·527019
558	311364	173741112	23·6220236	8·232746	621	385641	239483061	24·9198716	8·531601
559	312481	174676879	23·6431808	8·237661	622	386884	240641848	24·9399278	8·536178
560	313600	175616000	23·6643191	8·242571	623	388129	241804367	24·9599679	8·540750
561	314721	176558481	23·6854386	8·247474	624	389376	242970624	24·9799920	8·545317
562	315844	177504328	23·7065392	8·252371	625	390625	244140625	25·0000000	8·549879
563	316969	178453547	23·7276210	8·257263	626	391876	245314376	25·0199920	8·554437
564	318096	179406144	23·7486842	8·262149	627	393129	246491883	25·0399681	8·558990
565	319225	180362125	23·7697286	8·267029	628	394384	247673152	25·0599282	8·563537
566	320356	181321496	23·7907545	8·271904	629	395641	248858189	25·0798724	8·568081
567	321489	182284263	23·8117618	8·276773	630	396900	250047000	25·0998008	8·572619

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
631	398161	251239591	25·1197134	8·577152	694	481636	334255384	26·3438797	8·853598
632	399424	252435968	25·1396102	8·581681	695	483025	335702375	26·3628527	8·857849
633	400689	253636137	25·1534913	8·586205	696	484416	337153536	26·3818119	8·862095
634	401956	254840104	25·1793566	8·590724	697	485809	338608873	26·4007576	8·866337
635	403225	256047875	25·1992063	8·595238	698	487204	340068392	26·4196896	8·870576
636	404496	257259456	25·2190404	8·599747	699	488601	341532099	26·4386081	8·874810
637	405769	258474853	25·2388585	8·604252	700	490000	343000000	26·4575131	8·879040
638	407044	259694072	25·2586619	8·608753	701	491401	344472101	26·4764046	8·883266
639	408321	260917119	25·2784493	8·613248	702	492804	345948408	26·4952826	8·887488
640	409600	262144000	25·2982213	8·617739	703	494209	347428927	26·5141472	8·891706
641	410881	263374721	25·3179778	8·622225	704	495616	348913664	26·5329983	8·895920
642	412164	264609288	25·3377189	8·626706	705	497025	350402625	26·5518361	8·900130
643	413449	265847707	25·3574447	8·631183	706	498436	351895816	26·5706605	8·904336
644	414736	267089984	25·3771551	8·635655	707	499849	353393243	26·5894716	8·908538
645	416025	268336125	25·3968502	8·640123	708	501264	354894912	26·6082694	8·912737
646	417316	269586136	25·4165301	8·644585	709	502681	356400829	26·6270539	8·916931
647	418609	270840023	25·4361947	8·649044	710	504100	357911000	26·6458252	8·921121
648	419904	272097792	25·4558441	8·653497	711	505521	359425431	26·6645833	8·925308
649	421201	273359449	25·4754784	8·657946	712	506944	360944128	26·6833281	8·929490
650	422500	274625000	25·4950976	8·662391	713	508369	362467097	26·7020598	8·933668
651	423801	275894451	25·5147016	8·666831	714	509796	363994344	26·7207784	8·937843
652	425104	277167808	25·5342907	8·671266	715	511225	365525875	26·7394839	8·942014
653	426409	278445077	25·5538647	8·675697	716	512656	367061696	26·7581763	8·946181
654	427716	279726264	25·5734237	8·680124	717	514089	368601813	26·7768557	8·950344
655	429025	281011375	25·5929678	8·684546	718	515524	370146232	26·7955220	8·954503
656	430336	282300416	25·6124969	8·688963	719	516961	371694959	26·8141754	8·958658
657	431649	283593393	25·6320112	8·693376	720	518400	373248000	26·8328157	8·962809
658	432964	284890312	25·6515107	8·697784	721	519841	374805361	26·8514432	8·966957
659	434281	286191179	25·6709953	8·702188	722	521284	376367048	26·8700577	8·971101
660	435600	287496000	25·6904652	8·706587	723	522729	377933067	26·8886593	8·975240
661	436921	288804781	25·7099203	8·710983	724	524176	379503424	26·9072481	8·979376
662	438244	290117528	25·7293607	8·715373	725	525625	381078125	26·9258240	8·983509
663	439569	291434247	25·7487864	8·719759	726	527076	382657176	26·9443872	8·987637
664	440896	292754944	25·7681975	8·724141	727	528529	384240583	26·9629375	8·991762
665	442225	294079625	25·7875939	8·728518	728	529984	385828352	26·9814751	8·995883
666	443556	295408296	25·8069758	8·732892	729	531441	387420489	27·0000000	9·000000
667	444889	296740963	25·8263431	8·737260	730	532900	389017000	27·0185122	9·004113
668	446224	298077632	25·8456960	8·741624	731	534361	390617891	27·0370117	9·008223
669	447561	299418309	25·8650343	8·745985	732	535824	392223168	27·0554985	9·012329
670	448900	300763000	25·8843582	8·750340	733	537289	393832837	27·0739727	9·016431
671	450241	302111711	25·9036677	8·754691	734	538756	395446904	27·0924344	9·020529
672	451584	303464448	25·9229628	8·759038	735	540225	397065375	27·1108834	9·024624
673	452929	304821217	25·9422435	8·763381	736	541696	398688256	27·1293199	9·028715
674	454276	306182024	25·9615100	8·767719	737	543169	400315553	27·1477439	9·032802
675	455625	307546875	25·9807621	8·772053	738	544644	401947272	27·1661554	9·036886
676	456976	308915776	26·0000000	8·776383	739	546121	403583419	27·1845544	9·040965
677	458329	310288733	26·0192237	8·780708	740	547600	405224000	27·2029410	9·045041
678	459684	311665752	26·0384331	8·785029	741	549081	406869021	27·2213152	9·049114
679	461041	313046839	26·0576284	8·789346	742	550564	408518488	27·2396769	9·053183
680	462400	314432000	26·0768096	8·793659	743	552049	410172407	27·2580263	9·057248
681	463761	315821241	26·0959767	8·797968	744	553536	411830784	27·2763634	9·061310
682	465124	317214568	26·1151297	8·802272	745	555025	413493625	27·2946881	9·065367
683	466489	318611987	26·1342687	8·806572	746	556516	415160936	27·3130006	9·069422
684	467856	320013504	26·1533937	8·810868	747	558009	416832723	27·3313007	9·073473
685	469225	321419125	26·1725047	8·815160	748	559504	418508992	27·3495887	9·077520
686	470596	322828856	26·1916017	8·819447	749	561001	420189749	27·3678644	9·081563
687	471969	324242703	26·2106848	8·823731	750	562500	421875000	27·3861279	9·085603
688	473344	325660672	26·2297541	8·828009	751	564001	423564751	27·4043792	9·089639
689	474721	327082769	26·2488095	8·832285	752	565504	425259008	27·4226184	9·093672
690	476100	328509000	26·2678511	8·836556	753	567009	426957777	27·4408455	9·097701
691	477481	329939371	26·2868789	8·840823	754	568516	428661064	27·4590604	9·101726
692	478864	331373888	26·3058929	8·845085	755	570025	430368875	27·4772633	9·105748
693	480249	332812557	26·3248932	8·849344	756	571536	432081216	27·4954542	9·109767

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
757	573049	433798093	27·5136330	9·113781	820	672400	551368000	28·6356421	9·359902
758	574564	435519512	27·5317998	9·117793	821	674041	553387661	28·6530976	9·363705
759	576081	437245479	27·5499546	9·121801	822	675684	555412248	28·6705424	9·367505
760	577600	438976000	27·5680975	9·125805	823	677329	557441767	28·6879766	9·371302
761	579121	440711081	27·5862284	9·129806	824	678976	559476224	28·7054002	9·375096
762	580644	442450728	27·6043475	9·133803	825	680625	561515625	28·7228132	9·378887
763	582169	444194947	27·6224546	9·137797	826	682276	563559976	28·7402157	9·382675
764	583696	445943744	27·6405499	9·141788	827	683929	565609283	28·7576077	9·386460
765	585225	447697125	27·6586334	9·145774	828	685584	567363552	28·7749891	9·390242
766	586756	449455096	27·6767050	9·149757	829	687241	569722789	28·7923601	9·394020
767	588289	451217663	27·6947648	9·153737	830	688900	571787000	28·8097206	9·397796
768	589824	452984832	27·7128129	9·157714	831	690561	573856191	28·8270706	9·401569
769	591361	454756609	27·7308492	9·161686	832	692224	575930368	28·8444102	9·405339
770	592900	456533000	27·7488739	9·165656	833	693889	578009537	28·8617394	9·409105
771	594441	458314011	27·7668868	9·169622	834	695556	580093704	28·8790582	9·412869
772	595984	460099648	27·7848880	9·173585	835	697225	582182875	28·8963666	9·416630
773	597529	461889917	27·8028775	9·177544	836	698896	584277056	28·9236646	9·420387
774	599076	463684824	27·8208555	9·181500	837	700569	586376253	28·9309523	9·424142
775	600625	465484375	27·8388218	9·185443	838	702244	588480472	28·9482297	9·427894
776	602176	467288576	27·8567766	9·189402	839	703921	590589719	28·9654967	9·431642
777	603729	469097433	27·8747197	9·193347	840	705600	592704000	28·9827535	9·435388
778	605284	470910952	27·8926514	9·197289	841	707281	594823321	29·0000000	9·439131
779	606841	472729139	27·9105715	9·201229	842	708964	596947688	29·0172363	9·442870
780	608400	474552000	27·9284801	9·205164	843	710649	599077107	29·0344623	9·446607
781	609961	476379541	27·9463772	9·209096	844	712336	601211584	29·0516781	9·450341
782	611524	478211768	27·9642629	9·213025	845	714025	603351125	29·0688837	9·454072
783	613089	480048687	27·9821372	9·216950	846	715716	605495736	29·0860791	9·457800
784	614656	481890304	28·0000000	9·220873	847	717409	607645423	29·1032644	9·461525
785	616225	483736625	28·0178515	9·224791	848	719104	609800192	29·1204396	9·465247
786	617796	485587656	28·0356915	9·228707	849	720801	611960049	29·1376046	9·468966
787	619369	487443403	28·0535203	9·232619	850	722500	614125000	29·1547595	9·472682
788	620944	489303872	28·0713377	9·237528	851	724201	616295051	29·1719043	9·476395
789	622521	491169069	28·0891438	9·240433	852	725904	618470208	29·1890390	9·480106
790	624100	493039000	28·1069386	9·244335	853	727609	620650477	29·2061637	9·483813
791	625681	494913671	28·1247222	9·248234	854	729316	622835864	29·2232784	9·487518
792	627264	496793088	28·1424946	9·252130	855	731025	625026375	29·2403830	9·491220
793	628849	498677257	28·1602557	9·256022	856	732736	627222016	29·2574777	9·494919
794	630436	500566184	28·1780056	9·259911	857	734449	629422793	29·2745623	9·498615
795	632025	502459875	28·1957444	9·263797	858	736164	631628712	29·2916370	9·502308
796	633616	504358336	28·2134720	9·267680	859	737881	633839779	29·3087018	9·505998
797	635209	506261573	28·2311884	9·271559	860	739600	636056000	29·3257566	9·509685
798	636804	508169592	28·2488938	9·275435	861	741321	638277381	29·3428015	9·513370
799	638401	510082399	28·2665881	9·279308	862	743044	640503928	29·3598365	9·517051
800	640000	512000000	28·2842712	9·283178	863	744769	642735647	29·3768616	9·520730
801	641601	513922401	28·3019434	9·287044	864	746496	644972544	29·3938769	9·524406
802	643204	515849608	28·3196045	9·290907	865	748225	647214625	29·4108823	9·528079
803	644809	517781627	28·3372546	9·294767	866	749956	649461896	29·4278779	9·531749
804	646416	519718464	28·3548938	9·298624	867	751689	651714363	29·4448637	9·535417
805	648025	521660125	28·3725219	9·302477	868	753424	653972032	29·4618397	9·539082
806	649636	523606616	28·3901391	9·306328	869	755161	656234909	29·4788059	9·542744
807	651249	525557943	28·4077454	9·310175	870	756900	658503000	29·4957624	9·546403
808	652864	527514112	28·4253408	9·314019	871	758641	660776311	29·5127091	9·550059
809	654481	529475129	28·4429253	9·317860	872	760384	663054848	29·5296461	9·553712
810	656100	531441000	28·4604989	9·321697	873	762129	665338617	29·5465734	9·557363
811	657721	533411731	28·4780617	9·325532	874	763876	667627624	29·5634910	9·561011
812	659344	535387328	28·4956137	9·329363	875	765625	669921875	29·5803989	9·564656
813	660969	537367797	28·5131549	9·333192	876	767376	672221376	29·5972972	9·568298
814	662596	539353144	28·5306852	9·337017	877	769129	674526133	29·6141858	9·571938
815	664225	541343375	28·5482048	9·340838	878	770884	676836152	29·6310648	9·575574
816	665856	543338496	28·5657137	9·344657	879	772641	679151439	29·6479325	9·579208
817	667489	545338513	28·5832119	9·348473	880	774400	681472000	29·6647939	9·582840
818	669124	547343432	28·6006993	9·352286	881	776161	683797841	29·6816442	9·586468
819	670761	549353259	28·6181760	9·356095	882	777924	686128968	29·6984848	9·590094

No.	Square.	Cube.	Sq. Root.	CubeRoot	No.	Square.	Cube.	Sq. Root.	CubeRoot
883	779689	688465387	29·7153159	9·593716	942	887364	835896888	30·6920185	9·802804
884	781456	690807104	29·7321375	9·597337	943	889249	838561807	30·7083051	9·806271
885	783225	693154125	29·7489496	9·600955	944	891136	841232384	30·7245830	9·809736
886	784996	695506456	29·7657521	9·604570	945	893025	843908625	30·7408523	9·813199
887	786769	697864103	29·7825452	9·608182	946	894916	846590536	30·7571130	9·816659
888	788544	700227072	29·7993289	9·611791	947	896809	849278123	30·7733651	9·820117
889	790321	702595369	29·8161030	9·615398	948	898704	851971392	30·7896086	9·823572
890	792100	704969000	29·8328678	9·619002	949	900601	854670349	30·8058436	9·827025
891	793881	707347971	29·8496231	9·622603	950	902500	857375000	30·8220700	9·830476
892	795664	709732288	29·8663690	9·626201	951	904401	860085351	30·8382879	9·833924
893	797449	712121957	29·8831056	9·629797	952	906304	862801408	30·8544972	9·837369
894	799236	714516984	29·8998328	9·633390	953	908209	865523177	30·8706981	9·840813
895	801025	716917375	29·9165506	9·636981	954	910116	868250664	30·8868904	9·844254
896	802816	719323136	29·9332591	9·640569	955	912025	870983875	30·9030743	9·847692
897	804609	721734273	29·9499583	9·644154	956	913936	873722816	30·9192497	9·851128
898	806404	724150792	29·9666481	9·647737	957	915849	876467493	30·9354166	9·854562
899	808201	726572699	29·9833287	9·651317	958	917764	879217912	30·9515751	9·857993
900	810000	729000000	30·0000000	9·654894	959	919681	881974079	30·9677251	9·861422
901	811801	731431701	30·0166620	9·658468	960	921600	884736000	30·9838668	9·864848
902	813604	733870808	30·0333148	9·662040	961	923521	887503681	31·0000000	9·868272
903	815409	736314327	30·0499584	9·665609	962	925444	890277128	31·0161248	9·871694
904	817216	738763264	30·0665928	9·669176	963	927369	893056347	31·0322413	9·875113
905	819025	741217625	30·0832179	9·672740	964	929296	895841344	31·0483494	9·878530
906	820836	743677416	30·0998339	9·676302	965	931225	898632125	31·0644491	9·881945
907	822649	746142643	30·1164407	9·679860	966	933156	901428696	31·0805405	9·885357
908	824464	748613312	30·1330383	9·683416	967	935089	904231063	31·0966236	9·888767
909	826281	751089429	30·1496269	9·686970	968	937024	907039232	31·1126984	9·892175
910	828100	753571000	30·1662063	9·690521	969	938961	909853209	31·1287648	9·895580
911	829921	756058031	30·1827765	9·694069	970	940900	912673000	31·1448230	9·898983
912	831744	758550528	30·1993377	9·697615	971	942841	915498611	31·1608729	9·902383
913	833569	761048497	30·2158899	9·701158	972	944784	918330048	31·1769145	9·905782
914	835396	763551944	30·2324329	9·704699	973	946729	921167317	31·1929479	9·909178
915	837225	766060875	30·2489669	9·708237	974	948676	924010424	31·2089731	9·912571
916	839056	768575296	30·2654919	9·711772	975	950625	926859375	31·2249900	9·915962
917	840889	771095213	30·2820079	9·715305	976	952576	929714176	31·2409987	9·919351
918	842724	773620632	30·2985148	9·718835	977	954529	932574833	31·2569992	9·922738
919	844561	776151559	30·3150128	9·722363	978	956484	935441352	31·2729915	9·926122
920	846400	778688000	30·3315018	9·725888	979	958441	938313739	31·2889757	9·929504
921	848241	781229961	30·3479818	9·729411	980	960400	941192000	31·3049517	9·932884
922	850084	783777448	30·3644529	9·732931	981	962361	944076141	31·3209195	9·936261
923	851929	786330467	30·3809151	9·736448	982	964324	946966168	31·3368792	9·939636
924	853776	788889024	30·3973683	9·739963	983	966289	949862087	31·3528308	9·943009
925	855625	791453125	30·4138127	9·743476	984	968256	952763904	31·3687743	9·946380
926	857476	794022776	30·4302481	9·746986	985	970225	955671625	31·3847097	9·949748
927	859329	796597983	30·4466747	9·750493	986	972196	958585256	31·4006369	9·953114
928	861184	799178752	30·4630924	9·753998	987	974169	961504803	31·4165561	9·956477
929	863041	801765089	30·4795013	9·757500	988	976144	964430272	31·4324673	9·959839
930	864900	804357000	30·4959014	9·761000	989	978121	967361669	31·4483704	9·963198
931	866761	806954491	30·5122926	9·764497	990	980100	970299000	31·4642654	9·966555
932	868624	809557568	30·5286750	9·767992	991	982081	973242271	31·4801525	9·969909
933	870489	812166237	30·5450487	9·771484	992	984064	976191488	31·4960315	9·973262
934	872356	814780504	30·5614136	9·774974	993	986049	979146657	31·5119025	9·976612
935	874225	817400375	30·5777697	9·778462	994	988036	982107784	31·5277655	9·979960
936	876096	820025856	30·5941171	9·782946	995	990025	985074875	31·5436206	9·983305
937	877969	822656953	30·6104557	9·785429	996	992016	988047936	31·5594677	9·986649
938	879844	825293672	30·6267857	9·788909	997	994009	991026973	31·5753068	9·989990
939	881721	827936019	30·6431069	9·792386	998	996004	994011992	31·5911380	9·993329
940	883600	830584000	30·6594194	9·795861	999	998001	997002999	31·6069613	9·996666
941	885481	833237621	30·6757233	9·799334	1000	1000000	1000000000	31·6227766	10·000000

ANSWERS TO MISCELLANEOUS EXERCISES.

Exercise 8.

2. Sixty-seven trillions eight hundred and forty-five billions three hundred and ninety-eight millions six hundred and seventy-eight thousand nine hundred and four.
   Five quadrillions nine hundred trillions seven hundred and four billions sixty millions forty thousands, and sixty thousand six hundred and four *hundredths of millionths.*
3. M$\overline{\mathrm{V}}$DCCLXIX.
4. 429860000.
5. \$67·31$\frac{1}{4}$.
6. 77991.
7. 605000070016·000009.
8. 46978900.
10. 69·800463.
11. ·8439.
12. 678900000.
13. 604329860000000.
14. 1000001000001001·000000000001.
15. ·0007609.
16. Ninety trillions eight hundred and seven billions sixty millions five hundred and four thousand and thirty.
    Four quintillions four quadrillions forty trillions four hundred billions sixty thousand four hundred and thirty-two, and one trillion ten billion two hundred and three million forty thousand five hundred and six *hundredths of trillionths.*
18. 77$\frac{1}{3}$ cords.
19. 717 cords 91 cubic feet.
20. DCCXVIII, DCXIV, CDXCIX, CMXCIX, $\overline{\mathrm{V}}$MMMDCXLIII, $\overline{\mathrm{XCV}}$MCXLIX, $\overline{\mathrm{CLX}}$MMMCMLXXXVI, $\overline{\mathrm{CDXL}}$M$\overline{\mathrm{V}}$CDXLIV.
21. 333, 1989, and 1000001.
25. \$3·75$\frac{5}{12}$, \$24·58$\frac{1}{3}$, 71$\frac{2}{3}$, and \$757·47$\frac{1}{12}$.

Exercise 17.

1. \$18029304.
2. \$13999999·73.
3. 36497318.
4. 35857536.
5. 27424500.
6. 271633.
7. 9504000.
8. 327040000.
9. 92438 lbs. 8 oz. 2 dr. 1 scr. 13 grs.
10. 1698728602536.
11. 78990 bushels.
12. \$64·97.
13. 9032 yds. 3 qrs. 2 ns.
14. 1037957601·5.
15. \$16444·9602.

Exercise 22.

1. \$34736·8421.
2. \$30634·9206.
3. 3308 dys. or 9 yrs. $20\frac{3}{4}$ dys.
4. \$32.
5. \$137.
6. \$108.
7. \$9.
8. \$29.
9. $429\frac{82}{99}$.
10. ·578 oz.
11. $50\frac{2}{3}$.
12. 250 lbs.
13. 10·157.
14. 2 bush. 1 pk. 1 gal. 2 qts. $1\frac{2}{11}$ pts.
15. $1898\frac{266}{303}$.
16. 267 days $7\frac{18335}{23327}$ hours.

Exercise 23.

1. 789641420714.
2. Sixty-seven millions eight hundred and thirteen thousand four hundred and twenty, and twenty-one million thirty thousand and forty-six billionths.
Seventy-two millions, and seventy-two billionths.
One billion one million and one hundred, and ten trillion ten million and one tenths of quadrillionths.
3. DCCIX, $M\overline{V}CCCLXXVI$, $M\overline{X}CMXCIX$, $\overline{LXXXV}MIV$, $\overline{MMMCMXLV}MMDXCVI$.
4. 53973 lbs.
5. £3 18s. $11\frac{3}{4}$d.
6. 10837 yrs. 119 days 2 hours.
7. $\$2919\cdot50\frac{5}{12}$.
8. \$123·77.
9. 52000600204З·000000005016.
10. 1 acre 1 rood 3 per. 4 yds. 5 ft. 11 in.
11. \$12268·30.
12. 54 years 19 weeks 3 days 13 hours 33 minutes.
13. 741000000, ·00741, 741000000, ·000000741, ·00000000741, ·00741, and 74·1.
14. ·0331632.
15. $475\frac{25}{341}$ hhds.
16. \$6750.
17. $11\frac{41}{54}$.
18. 58 acres.
19. \$0·501.
20. \$37.
21. 3 lbs. 0 oz. 14 dwt. $13\frac{1}{3}$ grs.
22. 29 acres 0 roods 21 per.
23. 14 yds.
24. 15 lbs. 4 oz. 1 dwt. 14 grs.
25. $\$3890\cdot38\frac{3}{4}$.
26. 1032694.
27. 16800.
28. \$360·15.
29. \$247·95.
30. \$132082.
31. 169·49.
32. $\$79\cdot99\frac{7}{12}$.
33. \$59·85.
34. $\$532\cdot12\frac{1}{2}$.
35. $\overline{\overline{\overline{C}}}\,\overline{\overline{CC}}\,\overline{CC}DCCIX$.
36. ·56218+.
37. $1869696969\cdot\dot{6}\dot{9}$.
38. \$1713·34.
39. \$21·1433.
40. $236\frac{408}{519}$.

Exercise 40.

1. $\$4688\cdot16\frac{7}{12}$.
2. 27536 miles 1 fur. 21 per. 0 yds. 1 ft. 6 in.
3. 96.
4. 500313 *octenary* and 20222133 *quinary*.
5. 1243994·98275.

6. $\overline{LXX}M\overline{X}CDXXIII$ and $\overline{CCXXX}M\overline{V}DLXVII$.
7. 277200.
8. See XLVIII Recapitulation. Sec. I., page 57.
9. 642762977065601·1.
10. ——
11. See Table, page 125.
12. $2689·51⅞.
13. 27.
14. See Recapitulation XLVIII page 57.
15. 742000000905000078014·0000087200011.
16. Seventy-one trillions three hundred billions one hundred millions two hundred thousand four hundred and one, and seventy thousand four hundred and two trillionths.

One hundred and thirty-four quadrillions nine hundred trillions one hundred and one billions one hundred thousand and one hundred, and two hundred million twenty thousand and two trillionths.

Four quadrillions seven hundred trillions twenty thousand and seven, and two hundred and seventy-eight hundredths of trillionths.

17. £2272 0s. 3¼d.
18. $2^5 \times 5^3 \times 3 \times 23$.
19. 87 ft. 1′ 1″ 3‴ 0⁗ 10′′′′′ 8′′′′′′ 10′′′′′′′ 10′′′′′′′′
20. ·011436.
21, 16383.
22. 4096.
23. 11 acres 3 rds. 7 per. 19 yds. 0 ft. 130 in.
24. 336960.
25. Child's share, $179·41$\frac{3}{11}$; woman's, $358·82$\frac{6}{11}$; man's $1794·12$\frac{8}{11}$.
26. 1023 and 512.
27. 99$\frac{472}{1057}$.
28. 48359·8979694.
29. 722487·0873859.
30. 65 lbs. 7 oz. 0 drs. 1 scr.
31. 1, 2, 4, 7, 8, 14, 19, 28, 38, 56, 76, 133, 152, 266, 532, 1064.
32. 82$\frac{59}{186}$ yards.

Exercise 63.

1. $\frac{3}{5}$, $\frac{21}{100}$, $\frac{1}{20}$, $\frac{9}{25}$, and $\frac{7}{400}$.
2. $\frac{2}{45}$.
3. $4·52$\frac{1}{33}$.
4. $\frac{83}{136}$.
5. Gave away $\frac{29}{40}$ and kept $\frac{11}{40}$.
6. 1$\frac{53}{75}$.
7. $212·99$\frac{19}{52}$.
8. Longer part 72 feet and shorter part 64 feet.
9. 1058$\frac{43}{120}$ acres ; $13219·68¾.
10. 14$\frac{81}{160}$ and $\frac{97}{604}$.
11. $134·15⅝.
12. $28387·06¼.
13. 311$\frac{37}{60}$ bushels.
14. 1 and 1$\frac{359}{2176}$.
15. 2$\frac{17}{63}$ bushels.
16. $\frac{1}{7}$.
17. $\frac{481}{720}$.
18. 5$\frac{7}{36}$ and 2$\frac{31}{60}$.
19. $1333·33⅓ or $\frac{1}{30}$ of the whole.

Exercise 77.

1. ·8.
2. 1·4445566778.
3. 4 days 17 hours 55 min. 30 sec.
4. $\frac{18469}{19980}$.
5. $156{\cdot}85981\dot{2}7009\dot{4}$.
6. $\cdot73915\dot{7}19\dot{6}$ of a mile.
7. 16 sq. ft. 1045$\frac{5}{63}$ inches.
8. 1 acre 3 roods 13 per. 22 yds.

9. $11\frac{19}{66}$ and $1\frac{3}{70}$.

10. $26\cdot7837\dot{4}2857\dot{1}$.

11. $71\cdot3\dot{6}19\dot{3}$.
12. 11·546 oz.
13. $75\frac{1}{3}$ yards.

14. $13\cdot51\dot{6}953\dot{3}$.
15. 3, 3, 1, 4, 1, and 9.

16. $476\cdot650\dot{2}811\dot{9}$.
17. 9.

Exercise 78.

2. 702000007030017; 0000000004000076.

3. $1017116666\cdot\dot{6}$.
4. $2\frac{2}{3}$.
5. $10\frac{3887}{55660}$.
6. 5044 bricks.
7. 111 sq. ft. $0'\ 9''\ 7'''\ 4''''\ 5'''''$ $5''''''$
8. $8\frac{1444}{5445}$.
9. 12225 bush 2pks 0 gal 2 qts.

10. 20790.
11. 1375t·12 and 2049151.
12. 66.
13. 1 day 23 hours 24 min. $34\frac{14}{25}$ seconds.
14. 19860 lbs. 2 oz. $9\frac{1}{2}$ drs.
15. \$158·75.
16. $\frac{8}{9}$, $\frac{16}{99}$, $\frac{2248}{2475}$, and $\frac{377}{83250}$.

17. 7040000, ·0000704, 704000000000, ·00000000704, ·0000704, 7·04.

18. $3\frac{5062}{16831}$.
19. Man's share=£66 0s. $4\frac{1}{2}$d.,
woman's =£33 0s. $2\frac{1}{4}$d.,
child's =£11 0s. $0\frac{3}{4}$d.,
20. $120\frac{519}{3080}$.
21. 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 27, 30, 36, 45, 50, 54, 60, 75, 81, 90, 100, 108, 135, 150, 162, 180, 225, 270, 300, 324, 405, 450, 540, 675, 810, 900, 1350, 1620, 2025, 2700, 4050, 8100.
22. 117.
23. Lunar month=29 days 12 hours 44 min. 3 seconds. Solar year=365 days 5 hours 48 min. 48 seconds.

24. $13450\frac{19}{79}$.
25. $134062\frac{1}{2}$ lbs. or $13406\frac{1}{4}$ gals.
26. \$295·59$\frac{7}{17}$.
27. $247\frac{7}{17}$.
28. $6\frac{69}{128}$.
29. ——
30. $2^8\times3\times5$.
31. 55045884 lines.
32. \$45·59.
33. \$90·96$\frac{31}{64}$.
34. 3·185988.
35. $2159\frac{3}{34}$.
36. \$21588·90.
37. \$142·8248.
38. 293.
39. $\frac{1470}{2310}$, $\frac{1848}{2310}$, $\frac{1970}{2310}$, $\frac{560}{2310}$, $\frac{1815}{2310}$, $\frac{1617}{2310}$, $\frac{1155}{2310}$.
40. \$103·35$\frac{1}{2}$.

Exercise 89.

1. 2:3.
2. \$479·30$\frac{5}{8}$.
3. ——

4. Greatest 21:27; least 9:13.

5. $57\cdot100\dot{5}5566187249\dot{3}$.

6. 53ee3$\frac{7737}{1987}$*duodenary*, 12014313$\frac{410042}{1033440}$*quinary*, and 76040$\frac{9972}{12574}$*undenary*.

7. 5·57052 oz.
8. 3 yds. 3 qrs. 0 na. $0\frac{11}{24}$ in.
9. $2962·70.
10. 1 bush. 2 pk. 0 gal. 1 qt.
11. 17 : 8; 88 : 176; 17 : 8 and 23 : 11; 6 : 7 and 88 : 176; 1173 : 616.
12. 39 per cent.
13. $\frac{359}{2436}$.
14. $10\frac{23}{100}$.
15. £2 1s. $2\frac{1}{4}$d. nearly.
16. $3\frac{1}{15}$ days.
17. $\frac{70}{[illegible]}$.
18. 52.
19. $50\frac{25}{32}$.
20. ·026856599989+.
21. ·0778.
22. 4·32958 miles.
23. 764876887 *nonary*; 1001111010100001100111010000 *binary*; 11146453021 *septenary*.
24. 188100.
25. $80\frac{100}{363}$.
26. 48.
27. $415\cdot\dot{4}7113780\dot{4}$.
28. $53·5966.
29. 1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, 18, 20, 21, 25, 28, 30, 35, 36, 42, 45, 50, 60, 63, 70, 75, 84, 90, 100, 105, 126, 140, 150, 175, 180, 210, 225, 252, 300, 315, 350, 420, 450, 525, 630, 700, 900, 1050, 1260, 1575, 2100, 3150, 3600.
30. $5·04.
31. Each man's share, $$325\cdot99\frac{133}{647}$; each woman's, $$88\cdot90\frac{440}{647}$; each child's, $$25\cdot40\frac{128}{647}$.
32. $12\frac{5}{6}$, $5\frac{8}{17}$, $2\frac{3}{17}$.
33. 3 yds. 2 ft. $8\frac{2}{3}$ in.
34. 104 : 5.
35. 71 miles 5 fur. 34 per. 3 yards.
36. $\frac{2}{5}$.
37. $2\frac{65}{322}$.
38. 70 goats.
39. 200.

EXERCISE 92.

1. 7020400000, 7·0204, 70·204, ·0000070204, 7020·4, and ·00000070204.
2. 6704866·561.
3. £399 19s. $5\frac{16641}{17920}$d.
4. $846\cdot37\dot{2}09576\dot{3}$.
5. 5 : 7; 9 : 13; 54 : 221.
6. $2070·3593.
7. They have none.
8. $27431·31¼.
9. $\frac{11}{18}$, $\frac{714285}{999999}$, $\frac{35}{32}$, and $\frac{67}{111}$.
10. $2\frac{61}{340}$.
11. $12\frac{5}{6}$ days.
12. 744916400000; 7·449164; ·00000000007449164; 7449·164; ·0007449164; 744916·4.
13. ——
14. Binary 63 and 32, Quaternary 4095 and 1024, Senary 46655 and 7776, Octenary 262143 and 32768, Duodenary 2985983 and 248832.
15. 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 96, 108, 144, 192, 216, 288, 432, 576, 864, 1728.
16. 720720.
17. $79\cdot78\dot{9}96667774885\dot{5}$.
18. $127·98.
19. 21·19117.

EXERCISE 165.

1. 7000090000019·00000004200006.
2. A, $1639·32$\frac{1}{3}$; B, $1528·21$\frac{1}{3}$; C, 1437.31\frac{1}{3}$; D, $1524·95.
3. 13$\frac{4}{9}$.
4. $1497803819·4444.
5. 83160.
6. 361 y'rs. 10 m'ths. 25 days.
7. 40·38.
8. 33943 lbs. 4 oz. 8 dwt. 14$\frac{1}{2}$ grs.
9. 2.
10. 129$\frac{3}{8}$.
11. 3.
12. 24.
13. A, $384·47; B, $291·07; C, $221·89.
14. 135$\frac{34}{35}$ lbs.
15. ·165229.
16. 530·00121$\dot{8}$6450$\dot{0}$.
17. $7854·29.
18. 26$\frac{2}{3}$.
19. 81000.
20. 5456640.
21. They have none.
22. A, $3492·06; B, $4761·91; C, $6746·03.
23. A, £167$\frac{18}{43}$; B, £139$\frac{28}{43}$; C, £93$\frac{1}{43}$.
24. 2$\frac{8}{19}$ hours.
25. $\overline{\text{LXXM}}\overline{\text{V}}$CMXXXVIII and $\overline{\overline{\text{XV}}\text{MMCDXCV}}$MMMDCLXXIX.
26. 1st gets 792 loaves; 2nd, 594; 3rd, 924.
27. 72, 18 and 54 lbs., or 24, 96, and 96 lbs. respectively.
28. $3725·764.
29. 24010·23.
30. $4803·5064.
31. 5739·29 yds. Gain 25$\frac{5}{7}$ per cent.
32. ——
33. $126·12.
34. $\bar{2}$·886057; 1·290035; 3·051153, 1·449735; 4·812913; $\bar{4}$·698970; 2·182129; 0·909217.
35. *t*8·*t*2.
36. 84 years.
37. 66·80578 times.
38. 22992700·$\dot{7}$299270$\dot{0}$.
39. $5·482.
40. $460·0034.
41. 5 yrs. 8 mos. 5 days.
42. Amount $1409·07. Compound Int. $595·36.
43. 10 months 18 days.
44. A, $571·9675; B, $554·8675; C, $535·6375; D, $493·5275; and E, $1078.
45. $1372·02898.
46. 1.
47. 11704272374$\frac{2437}{4435}$ *octenary.*
48. ·01 and ·$\dot{0}$1234567$\dot{9}$.
49. One quadrillion three hundred billions fifty million and six thousand, and seven hundred million eighty thousand and nine trillionths.

Seven trillions six hundred billions two hundred and ninety millions thirty-four thousand and seven, and sixty-seven millions four hundred thousand two hundred and nine quadrillionths.

50. 1296.
51. 33·395 years.

52. $7119\frac{2}{39}$.
53. 144.
54. $35\frac{5}{8}$.
55. $8\frac{4}{7}$ days.
56. $2469·71.
57. $4\frac{2}{13}$, $3\frac{1}{13}$, and $2\frac{4}{13}$.
58. Each man had 60; A caught 50, B 60, C 70.
59. 191 and 17763.
60. 44·997 years.
61. A, $$1556·95\frac{2}{3}$; B, $$116[illegible]·95\frac{2}{3}$; C, $$973·08\frac{1}{3}$.
62. 1, 2, 4, 1429, 2858, 5716.
63. $2\frac{6}{25}$.
64. Man's share = $$919·14\frac{6}{47}$,
Woman's = $$459·57\frac{3}{47}$,
and child's = $$153·19\frac{1}{47}$.
65. 24.
66. $21·03.
67. Greatest 9 : 16; least 10 : 19; comp. ratio 21 : 247.

68. 8·5318452.
69. ·019156118.

70. 2781·8488131566898 29957.
71. 157·036 feet.
72. 85 spirits, 35 water.
73. 422·32.
74. 70 and 14.

75. 223·82460585.
76. 5·32341.
77. 58 and 28.
78. 156240.
79. 30401.
80. $228\frac{4}{7}$: 1617.
81. 3 and $1\frac{1}{2}$, or 4 and $1\frac{1}{3}$, or 5 and $1\frac{1}{4}$, &c.
82. $\frac{187}{763}$.
83. $5\frac{5}{11}$ minutes past 1 o'clock.

84. 6·585461; 3·502675; $\bar{5}$·187521; $\bar{2}$·113509; 0·196295; 1·969276.

85. $4·314.
86. X $672 and Y $1120.
87. $\frac{4}{27}$.
88. 4321.
89. $18\frac{3}{4}$ lbs. at 4d.; $18\frac{3}{4}$ lbs. at 6d.; and $74\frac{3}{4}$ lbs. at 8d.
90. 10, 22, 26.

91. 1, $8\frac{4}{5}$, $16\frac{3}{5}$, $24\frac{2}{5}$, $32\frac{1}{5}$, 40.
92. 7.
93. Apple 2d., pear 3d.
94. $\frac{19}{48}$.
95. $275.
96. $124 and $1564.

97. 1100000000011·0000000011.
98. $3649·3932.

99. $2^6 \times 3^2 \times 7 \times 11$.
100. $28\frac{1}{3}$.

101. 117.
102. $62\frac{4}{7}$ gal., $83\frac{3}{7}$ gal., and 146 gal.

103. A, £194 16s. $1\frac{19}{77}$d.; B, £129 17s. $4\frac{64}{77}$d.; C, £97 8s. $0\frac{46}{77}$d.; D, £77 18s. $5\frac{33}{77}$d.

104. $1230·338.
105. 10 hours.
106. 41 years.
107. 4·629 days.
108. £4 16s.
109. $44\frac{4}{13}$.
110. 1422·2 lbs.

111. 1st, ·46 inches; 2nd, ·57 in.; 3rd, ·82 in.; 4th, 3·149 in.
112. 71·117.
113. $2019·651; $4871·803; $4815·805; $6467·739; $1825.
114. 1st 300 yrs; 2nd 56·827 yrs.

115. 1st, $920·20; 2nd. $2760·60; 3rd, $5521·20.
116. Paid each workman $$28·66\frac{2}{3}$; 1st company cleared $87\frac{3}{47}$ acres; 2d company, $77\frac{11}{47}$ acres; cost of clearing, $$8\frac{82}{495}$ per acre.

117. 15 and 11.
118. \$2340·00,
119. 182 days.
120. A, \$2180; B, \$1635; C, \$1308; D, \$1090.
121. $\frac{7}{9}$, $\frac{83}{99}$, $\frac{787}{999}$, $\frac{18666}{49995}$, $8\frac{11111}{18888}$.
122. $861\frac{47}{79}$ and $411\frac{363}{1024}$.
123. Sum £58 0s. $8\frac{21}{160}$d.; quotient 32414·56.
124. $491\frac{1}{160}$ yds.
125. \$214.
126. 1st 175 yrs.; 2nd 41·914 yrs.
127. $10\frac{10}{31}$ perches.
128. 111104.
129. 9, 27, 81, 243, 729, 2187, 6561.
130. $9\frac{1}{4}$.
131. 8·04 in. 9·534 in. 12·426 in. and 30 inches.
132. 51 of each, rem. £$1\frac{3}{8}$.
133. \$200.
134. 19 per cent.
135. \$1888·888.
136. 1s. 9d., 1s. 2d., and 7d.
137. A, \$25; B, \$25; C, \$50; D, \$100.
138. ·057.
139. $\frac{32}{767}$; $162\frac{22}{145}$; $1\frac{181}{175}$; $\frac{54}{563}$; 2308.
140. 96; $17\frac{3}{8}$.
141. \$$89\frac{18}{25}$; \$$107\frac{17}{25}$; \$$148\frac{13}{25}$; and \$$179\frac{9}{25}$.
142. \$15009·84.
143. $17\frac{2}{5}$, $32\frac{4}{5}$, $48\frac{1}{5}$, and $63\frac{3}{5}$; 35 and 85905.
144. $36\frac{1}{4}$ days.

15
34
18
1
54
~~8~~

Take the number indicating the radix of the scale add 1 to it multiply it by 3 if 10 & add 1 if not ~~subtract the~~ ~~any~~ pif figures

THE END.

2 5 1
6 1
2 5 1
1 5 0 6
558) 1 5 3 1 1
1 1 1 6
4 1 5 1
3 5 0 6

27

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